Supplemental Material to Perceptually-Based Compensation of Light Pollution in Display Systems
Analytic Formulation for Scattering

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1 Descattering

To apply our framework to perform descattering for immersive displays we need an input image, and we need to be able to simulate the observed image (including pollution) during the perceptual minimization. The challenge with descattering is that just evaluating the pollution term is a complex and computationally expensive process which, in the general case, involves solving the rendering equation [Kajiya 1986].

1.1 Scattering on Arbitrary Lambertian Screens

We focus on projections designed for large audiences, and therefore assume screens with uniform Lambertian reflection properties. Projecting an image induces a discretization of the screen into patches with areas $\mathbf{a}_i$, defined by the projected area of each image pixel i. The projected area of the whole image is simply $a = \sum_i \mathbf{a}_i = \|\mathbf{a}\|_1$. Here we express the projected image $\mathbf{x}$ as $\mathbf{b}^0$. Furthermore, we assume $\mathbf{b}^0$ explicitly specifies the direct-illumination radiosity of each patch i on the screen, and not the emitted radiance of the projector. Note that, we ignore the projector-to-screen form factor for simplicity of notation, but this could easily be incorporated.

In the case of Lambertian reflection, the indirect illumination can be expressed recursively in terms of the input image $\mathbf{b}^0$ using the classical recursive radiosity equation:

$$\mathbf{b} = \mathbf{b}^0 + \rho \mathbf{F} \mathbf{b}, \tag{1}$$

where ρ is the diffuse reflectivity of the screen and $\mathbf{F}$ is the matrix of patch-to-patch form factors defined as:

$$\mathbf{F}_{ij} = \frac{1}{\mathbf{a}_i \pi} \int_{\mathbf{a}_j} \int_{\mathbf{a}_i} \frac{\cos\theta_x \cos\theta_y}{\|x-y\|^2} V(x,y)\, d\mathbf{a}_i(x)\, d\mathbf{a}_j(y), \tag{2}$$

where x and y are points on patches i and j respectively, θ_x and θ_y are the angles between the surface normals and the line connecting x and y, and V is the binary visibility function.

It is well known that Equation 1 can be solved for the observed image $\mathbf{b}$ in terms of the input image $\mathbf{b}^0$ as:

$$\mathbf{b} = \mathbf{N}\mathbf{b}^0 \qquad \text{where} \qquad \mathbf{N} = (\mathbf{I} - \rho\mathbf{F})^{-1}. \tag{3}$$

For descattering, this is the general matrix form of the abstract observation function $\psi(\mathbf{x})$, where $\mathbf{b}^0$ is the projected image $\mathbf{x}$ and $\mathbf{b}$ is the observed image $\psi(\mathbf{x})$.

Since the form factor matrix only depends on the geometry, the entire matrix can be stored and inverted as a preprocess – requiring only a matrix multiplication during runtime to compute the observed image. This can be further accelerated by employing approximation techniques from recent precomputed radiance transfer [Ng et al. 2003] and direct-to-indirect transfer methods [Hašan et al. 2006].

1.2 Scattering in a Spherical Dome

For descattering, in this paper we are primarily interested in projections onto spherical domes, such as an IMAX Dome. In this context, we introduce a novel method for efficiently computing the observed image *analytically*.

The image observed by the audience, $\mathbf{b}$, is a combination of the direct-illumination radiosity, $\mathbf{b}^0$, and a pollution term:

$$\mathbf{b} = \mathbf{b}^0 + \mathbf{b}^+, \tag{4}$$

where $\mathbf{b}_i^+$ is the additive scattering pollution observed at each discrete patch i. To compute the total indirect radiosity $\mathbf{b}^+$, we start by first expressing the 1-bounce radiosity and then generalize to subsequent bounces.

1.2.1 The First Indirect Bounce

The first indirect bounce of illumination can be expressed in terms of the direct radiosity using the discrete 1-bounce radiosity equation:

$$\mathbf{b}_i^1 = \rho \sum_j^n \mathbf{F}_{ij} \mathbf{b}_j^0. \tag{5}$$

Our key observation for solving this efficiently is that, due to the unique geometric structure of a sphere, the patch-to-patch form factors, $\mathbf{F}_{ij}$, can be expressed analytically. To show this, we consider the integrand in the definition of the form factor and note that the triangle formed by any points x, y and the center of the sphere is isosceles (see Figure 1), hence $\theta_x = \theta_y$. Further nothing that visibility is always one for a sphere, and applying the law of cosines and a double-angle identity we obtain:

$$\frac{\cos\theta_x \cos\theta_y}{\|x-y\|^2} = \frac{\cos^2\theta}{\|x-y\|^2} = \frac{\cos^2\theta}{4r^2\cos^2\theta} = \frac{1}{4r^2}, \tag{6}$$

where r is the radius of the sphere. This insight allows us to express the complete patch-to-patch form factor analytically simply as:

$$\mathbf{F}_{ij} = \frac{1}{\mathbf{a}_i \pi} \int_{\mathbf{a}_j} \int_{\mathbf{a}_i} \frac{1}{4r^2}\, d\mathbf{a}_i(x)\, d\mathbf{a}_j(y) = \frac{\mathbf{a}_j}{4\pi r^2}. \tag{7}$$

Note that this is simply the ratio of a patch area to the surface area of a full sphere. With this simplification, the 1-bounce radiosity expression in Equation 5 becomes:

$$\mathbf{b}^1 = \frac{\rho}{4\pi r^2} (\mathbf{a} \cdot \mathbf{b}^0). \tag{8}$$

What is notable about Equation 8 is it shows that 1-bounce scattering due to any projected image on a sphere is **spatially constant**, and can be computed with a single dot product.

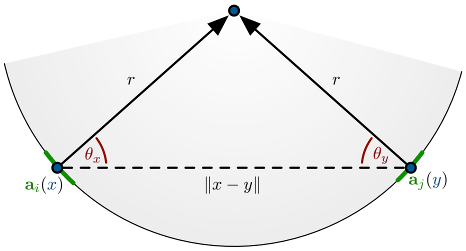

Figure 1: *The form factor geometry for a spherical dome.*

1.2.2 A Recurrence Relation for Subsequent Bounces

By recursively inserting subsequent bounces of light for $\mathbf{b}^0$ in the right-hand-side of Equation 8, we can obtain an expression for the 2-bounce radiosity. Repeating this process for an arbitrary number of bounces, m, results in a generalized expression of the form:

$$\mathbf{b}^m = \left(\frac{\rho}{4\pi r^2}\right)^m a^{m-1}(\mathbf{a}\cdot\mathbf{b}^0). \tag{9}$$

1.2.3 Total Indirect Illumination

The total indirect illumination, $\mathbf{b}^+$, is an infinite sum of all indirect bounces:

$$\mathbf{b}^+ = \mathbf{b}^1 + \cdots + \mathbf{b}^\infty = \frac{\rho}{4\pi r^2 - a\rho}(\mathbf{a}\cdot\mathbf{b}^0). \tag{10}$$

This is the pollution term, $\varphi(\mathbf{x})$, which our perceptual framework tries to minimize. Note that the total indirect illumination in a spherical dome is a single spatially uniform "ambient" *scalar*. Furthermore, Equation 10 makes it clear that all bounces of indirect illumination, for all pixels, can be computed using a single dot product in $O(n)$ time where n is the number of pixels in the image. Hence, this computation can be performed efficiently within the inner-loop of perceptual compensation, without down-sampling, even for high-resolution input images typical of IMAX Dome projection.

1.3 Subtractive Descattering

1.3.1 Subtractive Descattering for Arbitrary Screens

When negative values are not induced by a subtractive compensation algorithm, the subtractive compensation is the perceptual optimal solution. As subtractive compensation is in theory easier to compute, we can improve the efficiency of perceptual descattering by first checking if a subtractive compensation is possible.

Seitz et al. [2005] introduced the cancelation operator, which removes indirect scattering given an observed image. This is exactly the goal of descattering. We can express this in our notation by rewriting Equation 1 as:

$$\mathbf{b}^0 = \mathbf{b} - \rho\mathbf{F}\mathbf{b}. \tag{11}$$

Given a desired final observed radiosity $\mathbf{b}$, Equation 11 allows us to compute subtractive descattering of *all* bounces of indirect illumination for *arbitrary* screen configurations. The form factor matrix $\mathbf{F}$ can be precomputed, allowing us to perform subtractive descattering using a single matrix multiply during runtime. This was the technique suggested by Mukaigawa et al. [2006].

1.3.2 Analytic Subtractive Descattering for Spherical Domes

For the special case of a spherical dome we can obtain an efficient, analytic expression for subtractive descattering without the need to precompute the form factor matrix or to perform the matrix vector product in Equation 11. By exploiting the analytic expression for form factors from Equation 7 and inserting into Equation 11 we have:

$$\mathbf{b}^0 = \mathbf{b} - \frac{\rho}{4\pi r^2}(\mathbf{a}\cdot\mathbf{b}). \tag{12}$$

This simplifies the $n \times n$ matrix product into a single dot product and allows us to compensate for all bounces of scattering, at all patches of a spherical dome, analytically in just $O(n)$ time.

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