Subdivision Surfaces: A New Paradigm For Thin-Shell Finite-Element Analysis

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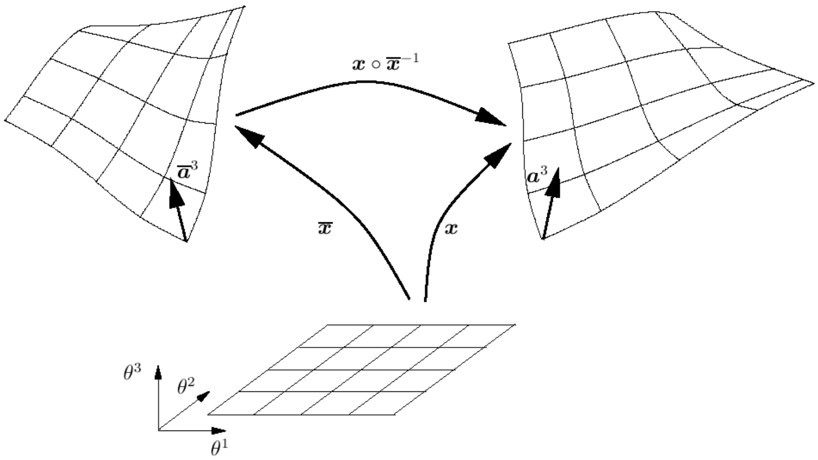
Overview

1. The problem with thin-shell finite-element analysis
2. Physics of thin-shells
3. Finite element discretization
4. Subdivision surfaces
5. Examples, convergence of method
6. Conclusions
7. My evaluation

1. Thin Shell Deformations The Problem

- Undeformed shell, deformed shell, finite element analysis

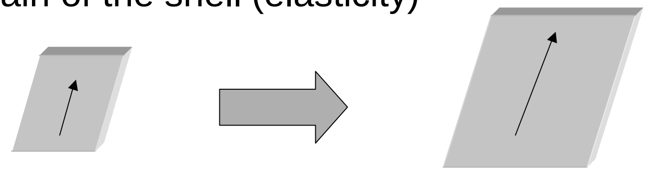


Thin-Shell Surface Problem

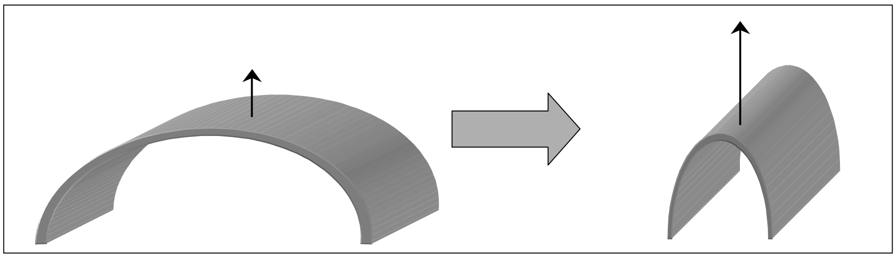
- Difficult to create C^1 continuity between Elements of the limit surface of a shell
- Shell must have a finite Kirchhoff-Love energy
- Usual methods include derivatives, lead to high order polynomials, difficult to calculate and physical limitations
- Purpose of paper: present a method that leads to the desired C^1 continuity.

2. Physics of deformation in Thin-Shell

- Energies involved in a deformation:
  - Strain of the shell (elasticity)



  - Variation of the bending of the shell



Physics of deformation in Thin-Shell

The physical formulas used to express this:

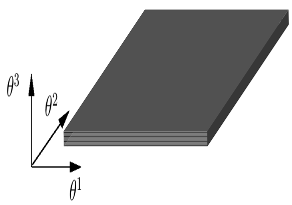
$$\alpha_{ij} = \tfrac{1}{2}(\vec{a}_i \vec{a}_j - \vec{\bar{a}}_i \vec{\bar{a}}_j) \quad \text{Strain tensor}$$

$$\beta_{\alpha\beta} = a_\alpha \frac{\partial a_3}{\partial \theta^\beta} - \bar{a}_\alpha \frac{\partial \bar{a}_3}{\partial \theta^\beta} \quad \text{Bending strain}$$

where $a, \bar{a}$ are basis vectors in undeformed and deformed shell respectively

$i, j \in \{1,2,3\}, \alpha, \beta \in \{1,2\}$

are the 3D components $(\theta^1, \theta^2, \theta^3)$



Physics of deformation in Thin-Shell

Since the deformed shell is the undeformed shell plus a deformation function (*linearization*), i.e.

$$x(\theta^1,\theta^2) = \bar{x}(\theta^1,\theta^2) + u(\theta^1,\theta^2)$$

where $x, \bar{x}$ are the coordinates in the deformed and the undeformed configuration,

u is displacement function

… the two deformation tensors can be expressed as a function of the not deformed coordinates and the displacement functions, which will be ideal for the finite-element analysis.

Equilibrium in the Deformation

- The energy density of the shell is a function of the previously defined α and β;
- The potential energy of the shell thus is

$$\Phi^{\text{int}}[u] = \int_{\Omega} W(\alpha,\beta)\,d\Omega$$

- The potential energy of the applied load is

$$\Phi^{ext}[u] = -\int_{\Omega} q \cdot u\,d\Omega - \int_{\partial\Omega} N \cdot u\,ds$$

where *q* are applied loads, *N* axial forces on boundary

Equilibrium in the Deformation

- In a stable configuration the sum of the potential energies must be minimal (physics)
- Potential energy: $\Phi[u] = \Phi^{\text{int}}[u] + \Phi^{\text{ext}}[u]$
  We minimize it according to Euler-Lagrange equations

$$\langle D\Phi[u], \delta u \rangle = \langle D\Phi^{\text{int}}[u], \delta u \rangle + \langle D\Phi^{\text{ext}}[u], \delta u \rangle = 0$$

- "Statement of the principle of virtual work": Actio=Reactio principle, force caused by shell deformation must be compensated by the force caused by the loads.

3. Finite-Element Discretization

- Build a mesh on the shell, choose base function
- The discretization leads to the expression:

$$K_h \cdot u_h = f_h$$

K is the energy of the shell

u is displacement field (array)

f is external force applied to the shell

- Done as sum over the elements
- *K* and *f* involve the evaluation of an integral
- Integrals can be computed with a quadrature rule
- Authors use a one-point quadrature rule, which is said to achive a sufficient precision for this analysis.

4. Subdivision Surfaces

- Construction of a smooth surface
- Done by repeated subdivision of a given mesh
- New nodes created at every subdivision
- Coordinates of nodes at step k+1 are computed as linear combination of nodes at step k
- Good choice of weights produce a smooth limit surface

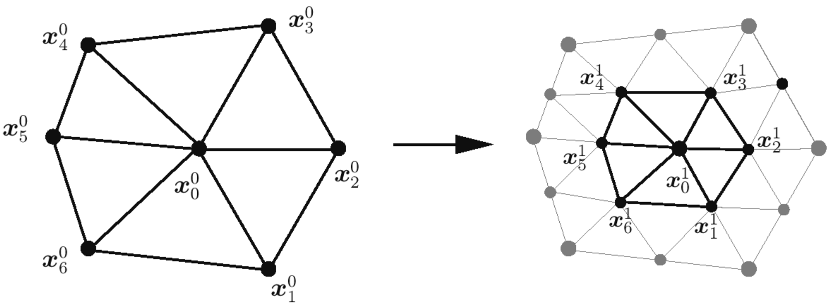
  (H^2 integrability, C^1 continuity)

Subdivision Schemes: Why?

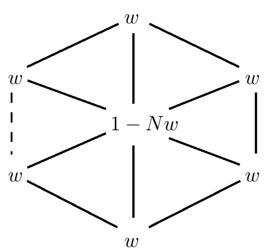
- 1D case easy to build C^n continuity through polynomial interpolation
- 2D surfaces: C^2 smoothness requires up to 6th order polynomials
- Difficulties arise when handling cross-patch smoothness
- Approximation scheme: C^2 continuity
  => subdivision surfaces are advantageous!

Subdivision Scheme: Loop

- Subdivision for triangulated meshes done with Loop's scheme, although every strategy could be used
- Leads to quadrisection of every triangle



Loop's Scheme

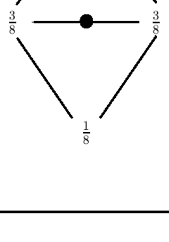


$$x_0^{k+1} = (1 - Nw)x_0^k + wx_1^k + \ldots + wx_N^k$$

where

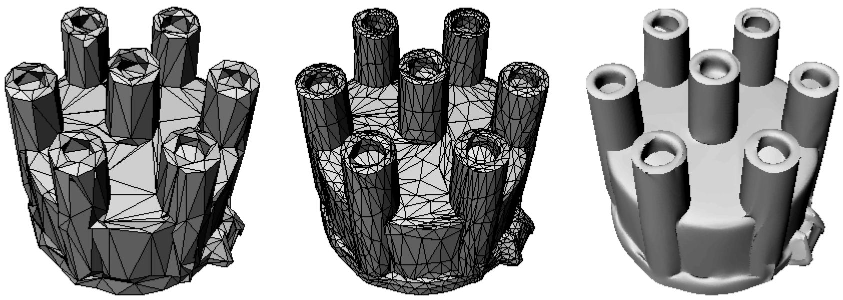
$$w = \tfrac{1}{N}[\tfrac{5}{8} - (\tfrac{3}{8} + \tfrac{1}{4}\cos(\tfrac{2\pi}{N})^2] \text{ (Loop)}$$

$$w = \begin{cases} \frac{3}{8N}, N > 3 \\ \frac{3}{16}, N = 3 \end{cases} \text{(Warren's Choice)}$$



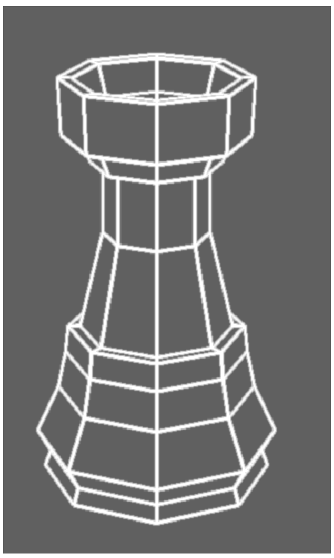
$$x_I^{k+1} = \frac{3x_0^k + x_{I-1}^k + 3x_I^k + x_{I+1}^k}{8}$$

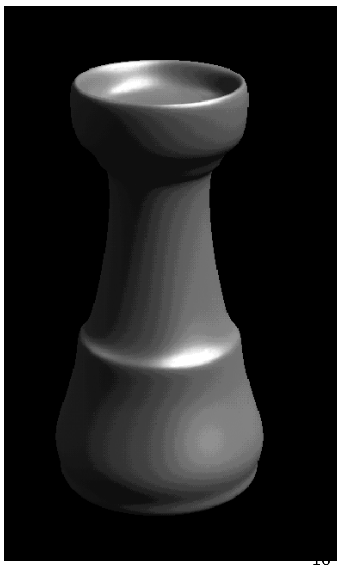
Loop’s Scheme: Examples



Loop on a Distributor Cap mesh

Loop’s Scheme: Examples





Convergence of limit surface

- The convergence of this method can be proven using a two vertex neighbourhood.
- Calculation of limit position of vertices using a one vertex neighbourhood:

Let $x^k = (x_0^k, x_1^k, ..., x_N^k)$, where N is valence of vertex.

We can express x^{k+1} as $x^{k+1} = S \cdot x^k$

S is a matrix expressing the Loop relationship.
Computation of limit configuration of vertices ($k \to \infty$)

$$x^\infty = (S)^\infty \cdot x^0, \qquad x^0 \text{ start configuration}$$

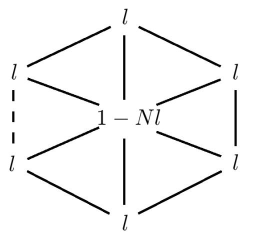
Convergence of limit surface

$$x^\infty = (S)^\infty \cdot x^0$$

Since $\lambda_0=1$, $\lambda_i \le 1$ for all $i \ne 0$, S^∞ converges to a limit.

Using eigenvalue/eigenvector decomposition, it can be shown that :

$$X^\infty = L_0 \cdot X^0$$



where

$$L_0 = (1 - N \cdot l, l, ..., l),$$

$$l = \frac{1}{\frac{3}{8w} + N}$$

Convergence of limit surface

- Similar consideration for all the limit values needed in the FE computation

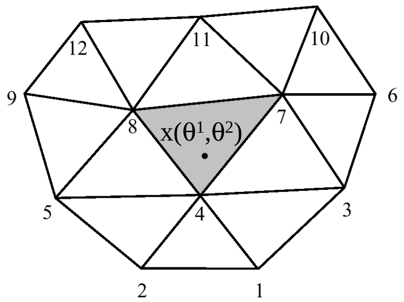
  (tangents on shell, surface normal)

- Advantage: computation of limit configuration of vertices and other primitives is possible at every step *k* of the refinement, i.e. when one has achieved the desired mesh subdivision.

- Convergence (regular patches, i.e. valence 6 at each vertex) to a quartic box Spline on every patch.

Evaluations in an Element

- We need to compute values of points and derivates inside an element (FE Analysis).
- For regular patches (vertices with valence 6):



$$x(\theta^1,\theta^2) = \sum_{i=1}^{12} N^i(\theta^1,\theta^2)\cdot x_i$$

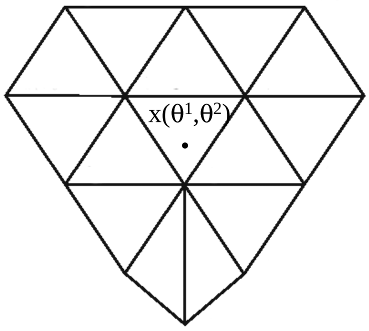
$$u_h(\theta^1,\theta^2) = \sum_{i=1}^{12} N^i(\theta^1,\theta^2)\cdot u_i$$

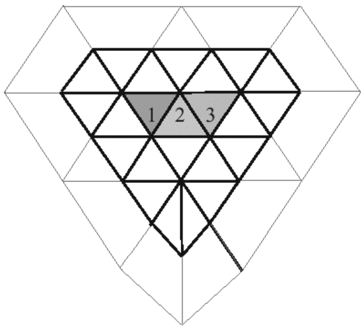
N spline shape functions

x,u vertex coordinates and displacements of neighbourhood

Evaluations in an Element

- Irregular patches: subdivide the element with Loop's scheme until the searched point is known to be in a regular sub-patch, then compute as before (with adapted parameters)





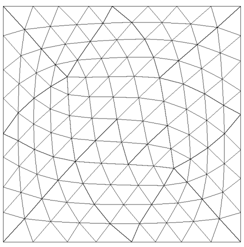
Implementation and computation

1. One subdivision step (Max one irregular vertex per patch)
2. Introduction of artificial nodes at boundary
3. Find 1-neighbourhood of vertices
4. Create local coordinates on irregular patches
5. Create stiffness matrix and force array
6. Introduce displacement boundaries
7. Solve system of equations (finite elements)
8. Compute limit position of nodes (sub. surfaces)

5. Examples and convergence

- The method is compared with two other approaches.
- A bound for a finite-element solution is known to exist.
- For the examples shown in the following, an analytical solution is known, thus we can analyse the “goodness” of this approach with the exact solution as well.

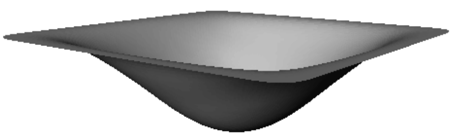
Rectangular Plate



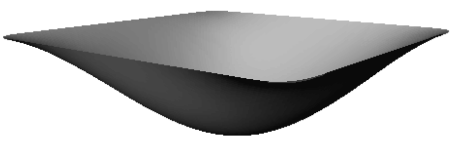
A typical mesh on such a plate

Irregular vertices are present

Uniform load on the shell

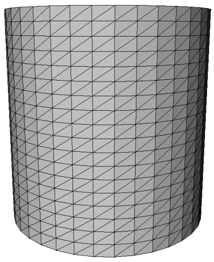


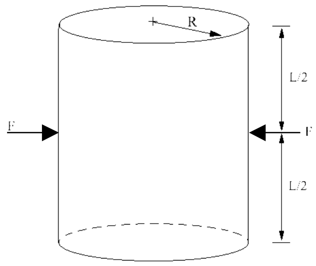
Clamped Boundary



Simply Supported Boundary

Pinched Cylinder

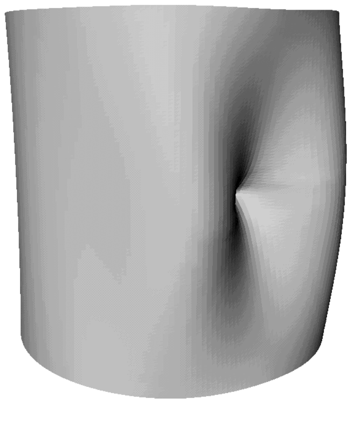


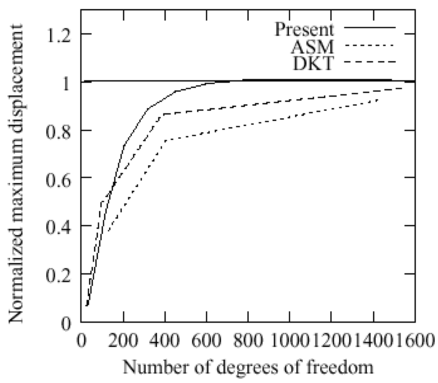


- Cylinder with unit loads applied on a mash point
- Loads diametrically opposed

- Due to subdivision scheme the loads spread over several points.
- The total weight is maintained

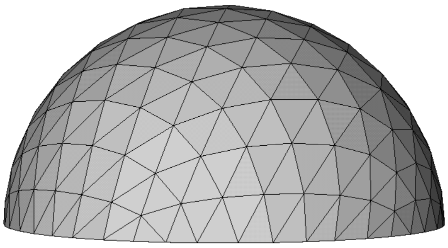
Pinched Cylinder



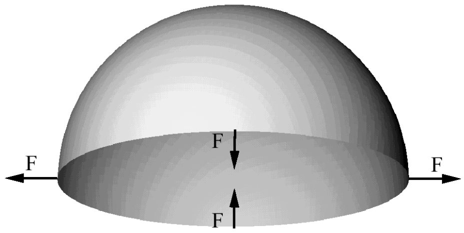


- Method converges to the optimal solution
- Convergence is faster than two other methods

Hemispherical Shell



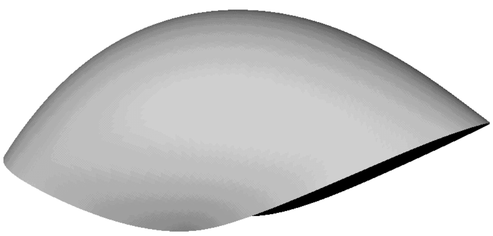
- Surface cannot be triangulated without irregular nodes
- Generalizations needed

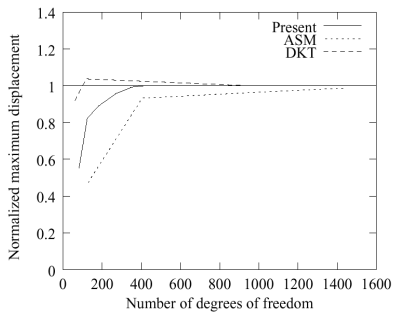


- Hard Test :

  Generic Box-Spline Approach not possible

Hemispherical Shell





- Surface converges optimally for this irregular mesh as well, even if standard approach is not possible
- Important that no parasitic strains appear.

6. Conclusions

- Use of subdivision surfaces for description of undeformed and deformed shell
- Method takes care of physical considerations (finite Kirchhoff-Love energy)
- Loop scheme: provable local convergence
- Smoothness between elements without using derivatives
- Finite element analysis on same mesh as subdivision (no additional triangulation error)

Conclusions

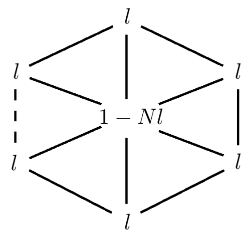
- Displacement field depends not only from element vertices, but from the 1-Neighbourhood as well
- Simple quadrature for finite-elements is sufficient
- Convergence is optimal in the finite element sense
- Method is applicable as well for other subdivision rules, not only for Loop scheme

7. My Opinion

- Easy to implement (easier than considering derivatives, mask existence, no particular data structures)
- No double meshes needed
- Respects physical laws

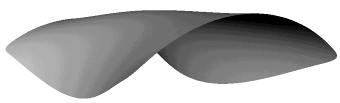
It would be interesting to see…

- How other schemes really behave
- Behaviour with not linearized Kinematics



$$\Phi[u] = \Phi^{\text{int}}[u] + \Phi^{\text{ext}}[u]$$

Questions ?



Kirchhoff-Love Energy

References

- Finite Elements for thin Shells and curved Members, Edited by D.G.Ashwell and R.H.Gallagher, John Wiley and sons, London 1976
- R. E. White: An introduction to the finite element method with applications to nonlinear problems
- Leif Kobblet: Subdivision Techniques for Curve and Surface Generation
- Siggraph Subdivision Course notes 2000: http://www.multires.caltech.edu/pubs/sig00notes.pdf http://www.mrl.nyu.edu/dzorin/sig00course/coursenotes00.pdf
- Integrated Modeling, Finite-Element Analysis, and Engineering Design for Thin-Shell Structures using Subdivision by Fehmi Cirak, Michael J. Scott, Erik K. Antonsson, Michael Ortiz, Peter Schröder http://www.multires.caltech.edu/pubs/design.pdf