

# 5

## Vector Field Visualization

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### Vector fields

A **static vector field**  $\mathbf{v}(\mathbf{x})$  is a vector-valued function of space. A **time-dependent vector field**  $\mathbf{v}(\mathbf{x}, t)$  depends also on time. In the case of **velocity fields**, the terms **steady** and **unsteady flow** are used.

The dimensions of  $\mathbf{x}$  and  $\mathbf{v}$  are equal, often 2 or 3, and we denote components by  $x, y, z$  and  $u, v, w$ :

$$\mathbf{x} = (x, y, z), \quad \mathbf{v} = (u, v, w)$$

Sometimes a vector field is defined on a surface  $\mathbf{x}(i, j)$ . The vector field is then a function of parameters and time:

$$\mathbf{v}(i, j, t)$$

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### Vector fields

An elementary visualization is to draw **arrows**

- at the data points (grid nodes or cell centers), or
- at a new (uniform) grid, for 3D fields often a 2D slice

Arrows can visualize:

- direction
- relative magnitude (when appropriately scaled)
- time dependency (when animated)

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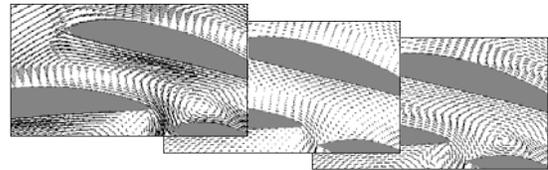
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### Vector fields

Problems of visualization with arrows:

- It is not clear whether arrows represent vector values at the start point or at the midpoint of the arrow
- Often no satisfactory scaling factor exists:
  - large scaling: Arrows occlude each other
  - small scaling: Direction is not recognizable in some regions
  - fixed length: Magnitude information is lost



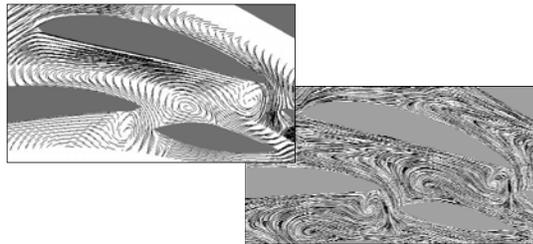
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### Vector fields

Streamline-based techniques for comparison: **streamlets** (short streamlines) and **LIC** (line integral convolution)



Magnitude information can be added by coloring or with animated texture.

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### Vector fields as ODEs

For simplicity, the vector field is now interpreted as a **velocity field**.

Then the field  $\mathbf{v}(\mathbf{x}, t)$  describes the connection between location and velocity of a (massless) particle.

It can equivalently be expressed as an **ordinary differential equation**

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t), t)$$

This ODE, together with an **initial condition**

$$\mathbf{x}(t_0) = \mathbf{x}_0,$$

is a so-called **initial value problem (IVP)**.

Its solution is the **integral curve**

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{v}(\mathbf{x}(\tau), \tau) d\tau$$

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The integral curve is a **pathline**, describing the **path** of a massless **particle** which was released at time  $t_0$  at position  $\mathbf{x}_0$ .

Remark:  $t < t_0$  is allowed.

For static fields, the ODE is **autonomous**:

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t))$$

and its integral curves

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{v}(\mathbf{x}(\tau)) d\tau$$

are called **field lines**, or (in the case of velocity fields) **streamlines**.

In **static** vector fields, pathlines and streamlines are **identical**.

In **time-dependent** vector fields, **instantaneous streamlines** can be computed from a "snapshot" at a fixed time  $T$  (being in a static vector field)

$$\mathbf{v}_T(\mathbf{x}) = \mathbf{v}(\mathbf{x}, T)$$

In practice, time-dependent fields are often given as a dataset per time step. Each dataset is then a snapshot.

### Streamlines, pathlines, streaklines, timelines

Besides **streamlines** and **pathlines**, two more types of lines can be obtained by integration: **streaklines** and **timelines**.

A **streakline** is obtained by continually releasing particles at a fixed location and taking a snapshot at a fixed time.

A **timeline** is obtained by simultaneously releasing particles densely on a **seed curve** and taking a snapshot at a fixed (later) time. This concept can be extended to **time surfaces**, obtained by releasing particles on a surface (e.g. rectangle or sphere).

Computing streaklines or timelines is more expensive than solving a single IVP.

### Streamlines, pathlines, streaklines, timelines

Algorithm for **streakline**:

- for time samples  $t_0, t_1, \dots, t_n$  solve the IVP

$$\dot{\mathbf{x}}_i(t) = \mathbf{v}(\mathbf{x}_i(t), t)$$

$$\mathbf{x}_i(t_i) = \mathbf{y}$$

- extract from each integral curve  $\mathbf{x}_i(t)$  the point  $\mathbf{x}_i(t_i)$
  - connect these points
- The result is a streakline for time  $t_n$ .

In the numerical computation the temporal interval must be **adaptively refined** if two successive particles diverge too much.

### Streamlines, pathlines, streaklines, timelines

Algorithm for **timeline**:

- for point samples  $\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_n$  on the seed curve solve the IVP

$$\dot{\mathbf{x}}_i(t) = \mathbf{v}(\mathbf{x}_i(t), t)$$

$$\mathbf{x}_i(t_0) = \mathbf{y}_i$$

- extract from the integral curve  $\mathbf{x}_i(t)$  the point  $\mathbf{x}_i(T)$
- connect these points

The result is a timeline for time  $T$ .

In the numerical computation the spatial interval must be **adaptively refined** if two neighbor particles diverge too much.

### Streamlines, pathlines, streaklines, timelines

Comparison of techniques:

(1) Pathlines:

- are physically meaningful
- allow comparison with experiment (observe marked particles)
- are well suited for dynamic visualization (of particles)

(2) Streamlines:

- are only geometrically, not physically meaningful
- are easiest to compute (no temporal interpolation, single IVP)
- are better suited for static visualization (prints)
- don't intersect (under reasonable assumptions)

*Streamlines, pathlines, streaklines, timelines*

(3) Streaklines:

- are physically meaningful
- allow comparison with experiment (dye injection)
- are well suited for static and dynamic visualization
- good choice for fast moving vortices
- can be approximated by set of disconnected particles

(4) Timelines:

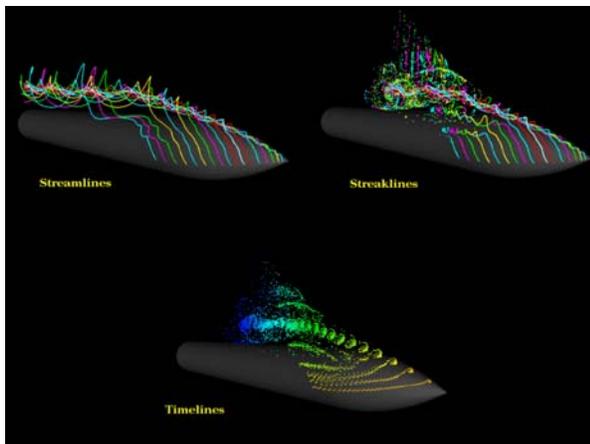
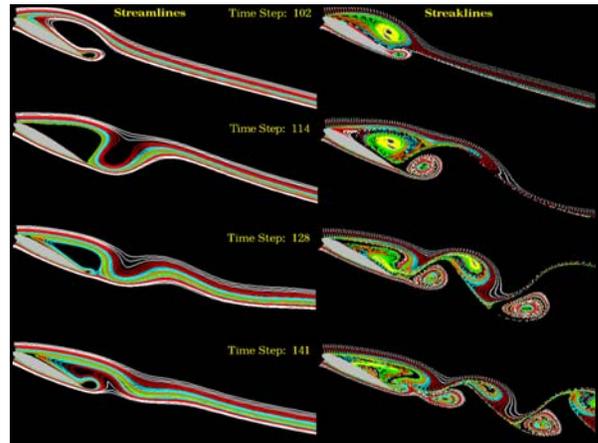
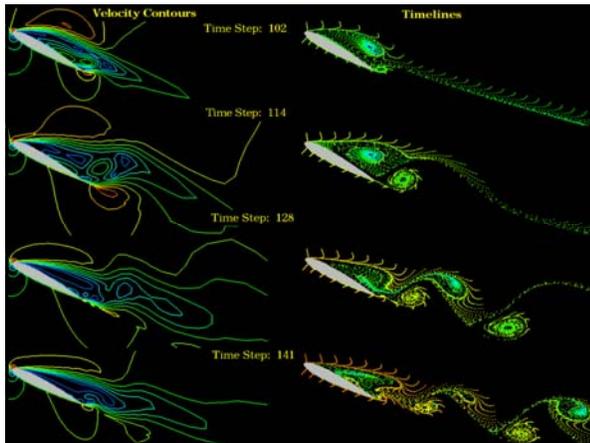
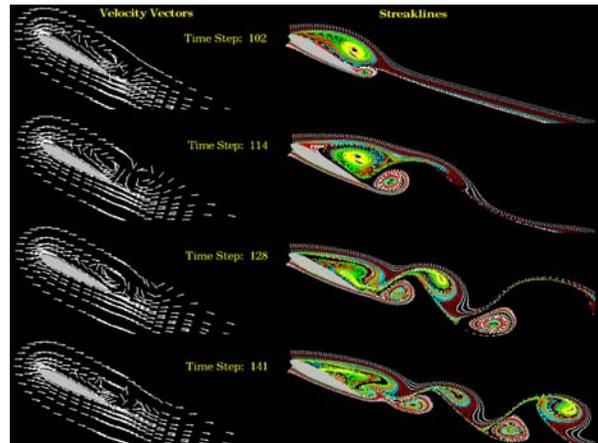
- are physically meaningful
- are well suited for static and dynamic visualization
- can be approximated by set of disconnected particles

Visual comparison of the techniques (from a NASA web page):

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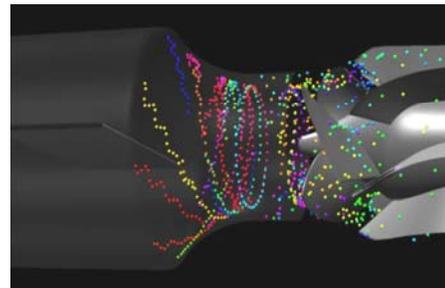
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*Streamlines, pathlines, streaklines, timelines*

Further example of (discrete) streaklines:



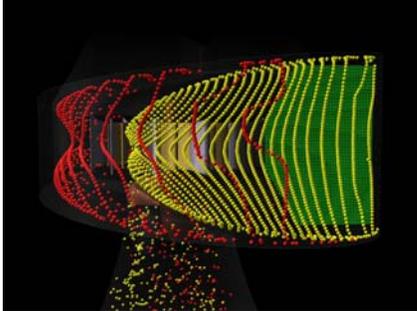
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Streamlines, pathlines, streaklines, timelines

Example of discrete time surfaces:



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Streamline integration

Outline of algorithm for numerical streamline integration (with obvious extension to pathlines):

Inputs:

- static vector field  $\mathbf{v}(\mathbf{x})$
- seed points with time of release  $(\mathbf{x}_0, t_0)$
- control parameters:
  - step size (temporal, spatial, or in local coordinates)
  - step count limit, time limit, etc.
  - order of integration scheme

Output:

- streamlines as "polylines", with possible attributes (interpolated field values, time, speed, arc length, etc.)

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Streamline integration

Preprocessing:

- set up search structure for point location
- for each seed point:
  - **global point location:** Given a point  $\mathbf{x}$ , find the cell containing  $\mathbf{x}$  and the local coordinates  $(\xi, \eta, \zeta)$  or if the grid is structured: find the computational space coordinates  $(i + \xi, j + \eta, k + \zeta)$
  - If  $\mathbf{x}$  is not found in a cell, remove seed point

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Streamline integration

Integration loop, for each seed point  $\mathbf{x}$ :

- interpolate  $\mathbf{v}$  trilinearly to local coordinates  $(\xi, \eta, \zeta)$
- do an integration step, producing a new point  $\mathbf{x}'$
- **incremental point location:** For position  $\mathbf{x}'$  find cell and local coordinates  $(\xi', \eta', \zeta')$  making use of information (coordinates, local coordinates, cell) of old point  $\mathbf{x}$

Termination criteria:

- grid boundary reached
- step count limit reached
- optional: velocity close to zero
- optional: time limit reached
- optional: arc length limit reached

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Streamline integration

Integration step: widely used integration methods:

- **Euler** (used only in special speed-optimized techniques, e.g. GPU-based texture advection)

$$\mathbf{x}_{new} = \mathbf{x} + \mathbf{v}(\mathbf{x}, t) \cdot \Delta t$$

- **Runge-Kutta**, 2<sup>nd</sup> or 4<sup>th</sup> order

Higher order than 4<sup>th</sup>?

- often too slow for visualization
- study (Yeung/Pope 1987) shows that, when using standard trilinear interpolation, **interpolation errors** dominate **integration errors**.

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Streamline integration

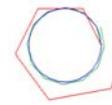
Example: Velocity field of **rigid rotation**

$$\mathbf{v}(\mathbf{x}) = (-\omega y, \omega x)$$

- $\mathbf{v}$  is **linear**, hence bilinear interpolation is **exact**
- observed errors are integration errors



Euler, small steps:  
 $\Delta t = 1/4, 1/8, 1/16$



RK2, larger steps:  
 $\Delta t = 1, 1/2, 1/4$

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Streamline integration

Choosing the step size:

Several options:  $\Delta t$

- fixed time step  $\Delta s$ 
  - used for animated particles
- fixed spatial step
  - time step derived from spatial step  $\Delta t = \Delta s / \|\mathbf{v}(\mathbf{x})\|$ , iteratively corrected
  - used for methods such as LIC
- adaptive
  - adapting to grid resolution, i.e. cell size, or
  - adapting to data variation (Runge-Kutta-Fehlberg method)
  - used for interactive viewing (with zooming)

The stencil walk algorithm

Incremental point location is nontrivial for curvilinear and unstructured grids.

Buning's stencil walk algorithm solves this problem.

Given:

- point with coordinates  $\mathbf{x}$
- cell (as three parameters  $(i, j, k)$  or as index  $c$ , resp.)
- local coordinates  $(\xi, \eta, \zeta)$
- coordinates of a new point  $\mathbf{x}'$

Wanted:

- new cell, as  $(i', j', k')$  or  $c'$  resp.
- new local coordinates  $(\xi', \eta', \zeta')$

The stencil walk algorithm

In a first phase the algorithm finds the cell containing  $\mathbf{x}'$  by doing iteratively:

- take the difference vector  $\Delta \mathbf{x} = \mathbf{x}' - \mathbf{x}$
- intersect the ray  $\mathbf{x} + t\Delta \mathbf{x}$  with the cell boundary, giving a  $t$  value
- if  $t \geq 1$  the point  $\mathbf{x}'$  lies in the current cell and iteration can be stopped
- otherwise move to the neighbor cell adjacent at the intersection point
- if no such cell exists terminate with failure
- set the cell centroid as the new  $\mathbf{x}$  for the next iteration

The stencil walk algorithm

The sub-problem of intersecting a ray with the cell boundary is solved as follows:

- Linearize the coordinate transform  $\varphi : (\xi, \eta, \zeta) \mapsto (x, y, z)$

in the point  $\mathbf{x} = (x, y, z)$ , i.e. compute the Jacobian

$$\mathbf{J} = \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}$$

- Using  $\mathbf{J}$  transform the difference vector  $\Delta \mathbf{x} = \mathbf{x}' - \mathbf{x}$  into the local coordinate frame of the cell

$$(\Delta \xi, \Delta \eta, \Delta \zeta) = \mathbf{J}^{-1} \Delta \mathbf{x}$$

The stencil walk algorithm

- Find the intersection of the ray

$$(\xi, \eta, \zeta) + t(\Delta \xi, \Delta \eta, \Delta \zeta)$$

with the cell boundary:

- having equations  $\xi, \eta, \zeta = 0$
- and for hex cell:  $\xi, \eta, \zeta = 1$
- resp. for tet cell:  $\xi + \eta + \zeta = 1$
- and inequalities:  $0 \leq \xi, \eta, \zeta \leq 1$

Due to linearization the point is not exact but in most cases (!) the correct neighbor cell is found.

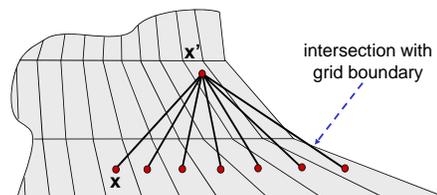
The stencil walk algorithm

Problem of original stencil walk algorithm:

If cells are sufficiently skewed, the algorithm can walk away from the target cell.

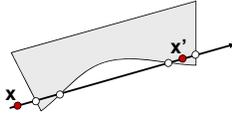
This happens also with correctly computed intersection points.

Example:



The stencil walk algorithm

- The problem can be solved with a **modification** of the algorithm:
- Keep ray  $(\mathbf{x}, \mathbf{x}')$  unchanged, i.e. follow the ray from  $\mathbf{x}$  to  $\mathbf{x}'$
  - new problem: cell faces of type quadrangle (nonplanar!) can be intersected **twice** by the ray!



- therefore, **exact** intersection points of the ray with the bilinear surface patches must be calculated

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The stencil walk algorithm

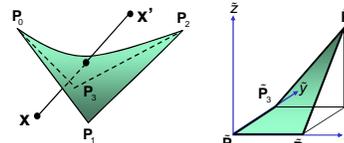
Exact intersection calculation:

- If the quadrangle is **nonplanar**, the four corners  $P_0, P_1, P_2, P_3$  can be mapped to

$$\tilde{P}_0 = (0,0,0), \tilde{P}_1 = (1,0,0), \tilde{P}_2 = (1,1,1), \tilde{P}_3 = (0,1,0)$$

by the **affine transformation**

$$\tilde{\mathbf{x}} = (\tilde{P}_1 | \tilde{P}_2 | \tilde{P}_3) (P_1 | P_2 | P_3)^{-1} (\mathbf{x} - P_0)$$



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The stencil walk algorithm

- The bilinear surface containing  $P'_0, P'_1, P'_2, P'_3$  is the **hyperbolic paraboloid**

$$z = xy$$

- Inserting the transformed view ray  $\tilde{\mathbf{x}} + t\Delta\tilde{\mathbf{x}}$  leads to a quadratic equation for  $t$ :

$$(\tilde{z} + t\Delta\tilde{z}) = (\tilde{x} + t\Delta\tilde{x})(\tilde{y} + t\Delta\tilde{y})$$

- If real solutions with  $0 < t < 1$  exist, the intersection points  $(\tilde{x}_i, \tilde{y}_i, \tilde{z}_i)$  are computed and transformed back

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The stencil walk algorithm

In the second phase the stencil walk algorithm computes the local coordinates of the point in the cell known to contain it.

The local coordinates are the inverse of the coordinate function

$$\varphi : (\xi, \eta, \zeta) \mapsto (x, y, z)$$

evaluated at the given point  $\mathbf{x} = (x, y, z)$

However, the trilinear function  $\varphi$  is a cubic polynomial and its inverse is a sixth-degree polynomial.

The problem is therefore solved with Newton's method.

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The stencil walk algorithm

Initialization:

- Let  $\mathbf{x}$  be the cell centroid
- The local coordinates  $(\xi, \eta, \zeta)$  are then:
  - for a hex cell: (0.5, 0.5, 0.5)
  - for a tet cell: (0.25, 0.25, 0.25)

Loop:

Repeat while the error  $\Delta\mathbf{x} = \mathbf{x}' - \mathbf{x}$  is above a given threshold:

- Compute the (vector) coefficients  $\mathbf{a}, \mathbf{b}, \dots$  of  $\varphi$ :

$$\varphi(\xi, \eta, \zeta) = \mathbf{a}\xi\eta\zeta + \mathbf{b}\xi\eta + \mathbf{c}\xi\zeta + \mathbf{d}\eta\zeta + \mathbf{e}\xi + \mathbf{f}\eta + \mathbf{g}\zeta + \mathbf{h}$$

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The stencil walk algorithm

- Evaluate the Jacobian in  $(\xi, \eta, \zeta)$ :

$$\mathbf{J} = \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)} = \begin{pmatrix} \frac{\partial\varphi}{\partial\xi} & \frac{\partial\varphi}{\partial\eta} & \frac{\partial\varphi}{\partial\zeta} \end{pmatrix} = (\mathbf{a}\eta\zeta + \mathbf{b}\eta + \mathbf{c}\zeta + \mathbf{e} \quad \dots \quad \dots)$$

- Transform the error vector  $\Delta\mathbf{x} = \mathbf{x}' - \mathbf{x}$  into local coordinates:

$$(\Delta\xi, \Delta\eta, \Delta\zeta) = \mathbf{J}^{-1}\Delta\mathbf{x}$$

- Apply the correction

$$(\xi, \eta, \zeta)_{\text{new}} = (\xi, \eta, \zeta) + (\Delta\xi, \Delta\eta, \Delta\zeta)$$

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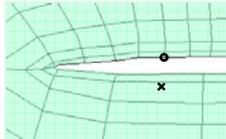
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### Global point location

Global point location is more expensive. Many methods trade in safety for efficiency. A few methods are:

- (1) Search for the point in every grid cell, using Newton's method. Hypothetic "brute force" method, safe.
- (2) Buning's method: Do incremental point location starting from a boundary cell. Problem: Node (o) nearest to given point (x) is not necessarily adjacent to the cell containing it. Furthermore, the straight line between the two points can leave the grid.



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### Global point location

Buning's method is safe if incremental search is repeated with a different boundary cell as long as the point is not found. Instead of using all boundary cells, a subset can be precomputed which guarantees to find all points within the grid.

- (3) Do incremental point location, starting a node near grid center. Simple method, safe only for **star-shaped** grids.
- (4) Efficient methods use a search structure (uniform grid, octree, kd-tree) for nodes or cell centers:
  - Point query not sufficient, need **range query**, with range determined by cell size.
  - Problem: cells (especially from CFD) can have extreme aspect ratios.

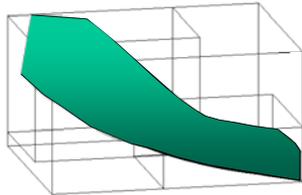
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### Global point location

- (5) Use a bounding box hierarchy for recursively subdivided grid. Efficient and safe method. Easy for structured grids. More pre-processing required for unstructured grids.



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### Computational space streamline integration

In structured grids, **point location** can be **avoided** by using a different approach:

Integration can be done in computational space  $\mathcal{C}$  instead of physical space  $\mathcal{P}$ .

Modification of the integration algorithm:

- **before** integration step:
  - Transform **velocity**  $\mathbf{v}(\mathbf{x})$  to  $\mathcal{C}$  by multiplying with  $\mathbf{J}^{-T}$
- **after** integration step:
  - (only if graphical output of this step is needed):
  - transform new **position**  $\mathbf{x} = (i + \xi, j + \eta, k + \zeta)$  to  $\mathcal{P}$  by trilinear interpolation.

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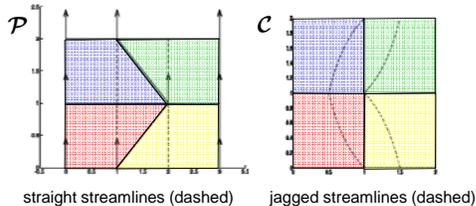
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### Computational space streamline integration

Main problem of integration in  $\mathcal{C}$ :

- coordinate function  $\varphi : (i + \xi, j + \eta, k + \zeta) \mapsto (x, y, z)$  is only  $C^0$  continuous at cell boundaries
- therefore  $\mathbf{J}$  is discontinuous

Example (Sadarjoen 1994): Four cells with constant velocity field.



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### Computational space streamline integration

Two sources of error:

- Integration steps across cell boundaries:
  - can be avoided by shortening such steps
- Use of a (precomputed) single transformed field vector per node:
  - can be fixed by transforming all eight vectors of a cell on the fly when entering a new cell.

The main advantage of integration in  $\mathcal{C}$  is algorithmic simplicity. If done correctly (avoiding above errors) it can be slower than integration in  $\mathcal{P}$ .

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### Skin friction lines

Velocity fields of fluid flow can have grid boundaries which are **walls**, i.e. solid material surfaces. At walls the velocity vector is usually zero as a result of the no-slip boundary condition.

Therefore, a derived vector field is often used, namely the **wall shear stress**:

$$\tau_w = \mu \frac{\partial \mathbf{v}_w}{\partial s}$$

obtained as the limit of the wall-parallel velocity component  $\mathbf{v}_w$  divided by the wall distance  $s$  and multiplied with the dynamic viscosity  $\mu$  (a material constant). This limit is typically nonzero except at isolated points.

Streamlines of  $\tau_w$  are called **skin friction lines**. They are an example of a vector field defined on a surface in 3-space.

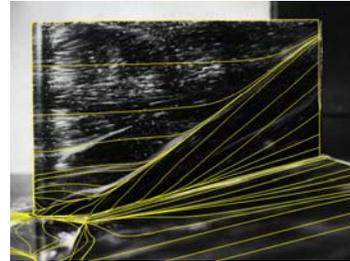
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### Skin friction lines

Example (Pagendarm and Walter 1994): Skin friction lines from numerical simulation superposed over experimental oil-flow pattern.



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### Streamline placement

Problem of visualization by streamlines:

- dependency on seed points
- density of streamlines can be largely inhomogeneous

Solution: **streamline placement**, i.e. automatic, optimized choice of seed points.

Method 1: **streamlets** (short streamline segments)

- length is proportional to velocity magnitude (obtained automatically by using fixed integration time)
- start with uniform grid and make spacing roughly even by locally adapting (displacing, inserting, removing) seeds

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### Streamline placement

Example of streamlets:



Discussion: Compared to arrows:

- additional curvature information
- no overlapping

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### Streamline placement

Method 2, Algorithm by Turk and Banks (for longer streamlines):

- Objective: Create a streamline image which when low-pass filtered has a uniform grey level
- Optimize: seed positions and integration lengths
- Operations:
  - delete, insert, move, lengthen, shorten
- Apply operations either randomly or based on oracles.



Seeds on regular grid vs. Turk/Banks method

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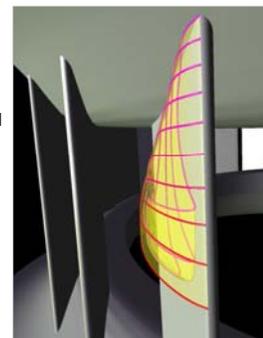
### Streamsurfaces

Definition of a **stream surface**:  
Union of streamlines seeded densely on a curve, e.g. straight line or circle.

Advantage for visualization:  
more structured, better spatial perception.

Naïve algorithm:

- start integration at discrete samples on the seed curve
- connect points of equal integration time, resulting in a quad mesh.



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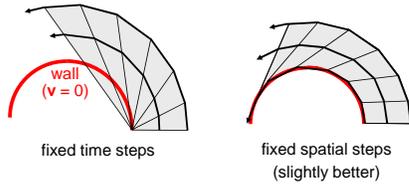
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Streamsurfaces

Problem: naive algorithm fails if streamlines diverge or grow at largely different speeds.

Example of failure: seed curve which extends to no-slip boundary:



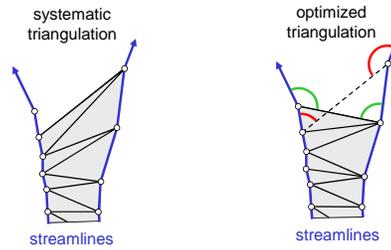
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Streamsurfaces

Hultquist's algorithm solves the problem of speed differences by **optimized triangulation**: Of two possible connections chose the one which is closer to orthogonal to both streamlines.



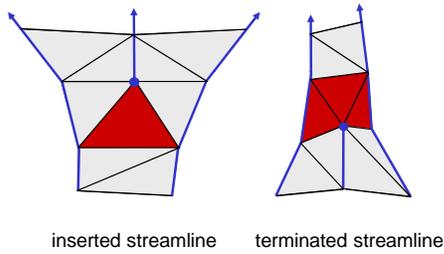
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Streamsurfaces

The problem of divergence or convergence is solved by **inserting** or **terminating** streamlines.



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