8

Vector Field Topology

Vector fields as ODEs

What are conditions for existence and uniqueness of streamlines?

- For the initial value problem

  $$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t)) \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

  a solution exists if the velocity field $\mathbf{v}(\mathbf{x})$ is continuous.
- The solution is unique if the field is Lipschitz-continuous, i.e. if there is a constant *M* such that

  $$\|\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x}')\| \le M \|\mathbf{x} - \mathbf{x}'\|$$

  for all $\mathbf{x}'$ in a neighborhood of **x**.

Vector fields as ODEs

Lipschitz-continuous is stronger than continuous (C^0) but weaker than continuously differentiable (C^1).

Important for scientific visualization:

- piecewise multilinear functions are Lipschitz-continuous
- in particular cellwise bi- or trilinear interpolation is Lipschitz-continuous

Consequence: Numerical vector fields do have unique streamlines, but analytic vector fields don't necessarily.

Vector fields as ODEs

Example: for the vector field

$$\mathbf{v}(\mathbf{x}) = (u(x,y), v(x,y)) = (1, 3y^{2/3})$$

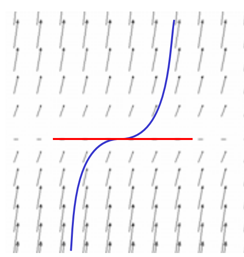
the initial value problem

$$\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t)) \qquad \mathbf{x}(0) = \mathbf{x}_0$$

has the two solutions

$$\mathbf{x}_{red}(t) = (x_0 + t, 0)$$

$$\mathbf{x}_{blue}(t) = (x_0 + t, t^3)$$



Both are streamlines seeded at the point $(x_0, 0)$.

Special streamlines

It is possible that a streamline $\mathbf{x}(t)$ maps two different times *t* and *t'* to the same point:

$$\mathbf{x}(t) = \mathbf{x}(t') = \mathbf{x}_1$$

There are two types of such special streamlines:

- stationary points: If $\mathbf{v}(\mathbf{x}_1) = 0$, then the streamline degenerates to a single point

  $$\mathbf{x}(t) = \mathbf{x}_1 \quad (t \in \mathbb{R})$$

- periodic orbits: If $\mathbf{v}(\mathbf{x}_1) \ne 0$, then the streamline is periodic:

  $$\mathbf{x}(t + kT) = \mathbf{x}(t) \quad (t \in \mathbb{R}, k \in \mathbb{Z})$$

All other streamlines are called regular streamlines.

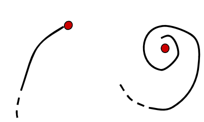
Special streamlines

Regular streamlines can converge to stationary points or periodic orbits, in either positive or negative time.

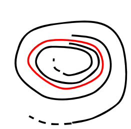
However, because of the uniqueness, a regular streamline cannot contain a stationary point or periodic orbit.

Examples: convergence to

- a stationary point



- a periodic orbit



Critical points

A stationary point $\mathbf{x}_c$ is called a critical point if the velocity gradient $\mathbf{J} = \nabla \mathbf{v}(\mathbf{x})$ at $\mathbf{x}_c$ is regular (is a non-singular matrix, has nonzero determinant).

Near a critical point, the field can be approximated by its linearization

$$\mathbf{v}(\mathbf{x}_c + \mathbf{x}) = \mathbf{J}\mathbf{x} + O(\mathbf{x}^2)$$

Properties of critical points:

- in a neighborhood, the field takes all possible directions
- critical points are isolated (as opposed to general stationary points, e.g. points on a no slip boundary)

Critical points

Critical points can have different types, depending on the eigenvalues of **J**, more precisely on the signs of the real parts of the eigenvalues.

We define an important subclass:

A critical point is called hyperbolic if all eigenvalues of **J** have nonzero real parts.

The main property of hyperbolic critical points is structural stability: Adding a small perturbation to **v**(**x**) does not change the topology of the nearby streamlines.

Critical points in 2D

Hyperbolic critical points in 2D can be classified as follows:

- two real eigenvalues:
  - both positive: node source
  - both negative: node sink
  - different signs: saddle
- two conjugate complex eigenvalues:
  - positive real parts: focus source
  - negative real parts: focus sink

Critical points in 2D

In 2D the eigenvalues are the zeros of

$$x^2 + px + q = 0$$

where p and q are the two invariants:

$$p = -\mathrm{trace}(\mathbf{J}) = -(\lambda_1 + \lambda_2)$$
$$q = \det(\mathbf{J}) = \lambda_1 \lambda_2$$

The eigenvalues are complex exactly if the discriminant

$$D = p^2 - 4q$$

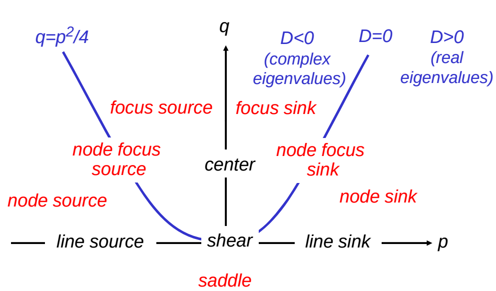
is negative.

It follows:

- critical point types depend on signs of p,q and D
- hyperbolic points have either $q < 0$, or $q > 0$ and $p \neq 0$

Critical points in 2D

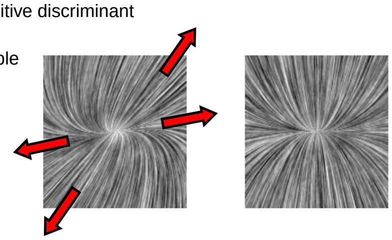
The p-q chart (hyperbolic types printed in red)



Node source

- positive trace
- positive determinant
- positive discriminant

Example

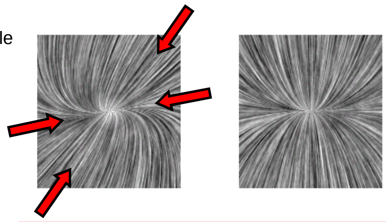


$$\mathbf{J} = \begin{pmatrix} 0.425 & 0.43125 \\ -0.1 & 1.075 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{A}$$

Node sink

- negative trace
- positive determinant
- positive discriminant

Example

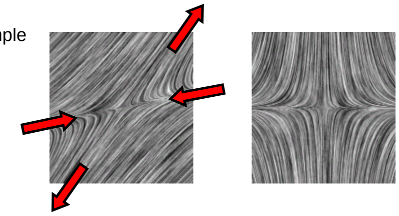


$$\mathbf{J} = \begin{pmatrix} -0.425 & -0.43125 \\ 0.1 & -1.075 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} -0.5 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{A}$$

Saddle

- any trace
- negative determinant
- positive discriminant

Example



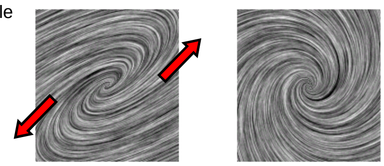
$$\mathbf{J} = \begin{pmatrix} -0.43375 & 1.07812 \\ -0.25 & 1.15 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} -0.25 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{A}$$

Focus source

- positive trace
- positive determinant
- negative discriminant

counter-clockwise if
$\partial v / \partial x - \partial u / \partial y > 0$

Example



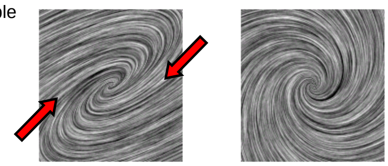
$$\mathbf{J} = \begin{pmatrix} 1.48 & -1.885 \\ 1.04 & -0.48 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 0.5 & -1 \\ 1 & 0.5 \end{pmatrix} \mathbf{A}$$

Focus sink

- negative trace
- positive determinant
- negative discriminant

counter-clockwise if
$\partial v / \partial x - \partial u / \partial y > 0$

Example



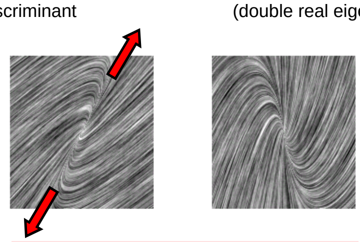
$$\mathbf{J} = \begin{pmatrix} -1.48 & 1.885 \\ -1.04 & 0.48 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} -0.5 & 1 \\ -1 & -0.5 \end{pmatrix} \mathbf{A}$$

Node focus source

- positive trace
- positive determinant
- zero discriminant

between node source
and focus source
(double real eigenvalue)

Example

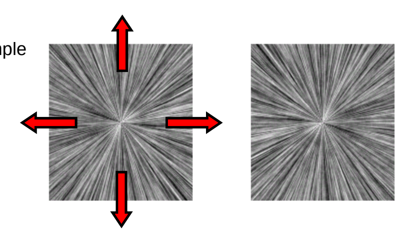


$$\mathbf{J} = \begin{pmatrix} 1.25 & -0.5625 \\ 1 & -0.25 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 0.5 & 0 \\ 1 & 0.5 \end{pmatrix} \mathbf{A}$$

Star source

Special case of node focus source: diagonal matrix

Example



$$\mathbf{J} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

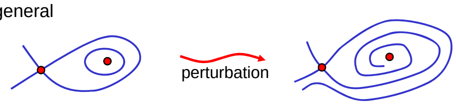
Nonhyperbolic critical points

If the eigenvalues have zero real parts but are nonzero (eigenvalues are purely imaginary), the critical point is the boundary case between focus source and focus sink.

This type of critical point is called a center.

Depending on the higher derivatives, it can behave as a source or as a sink.

Because a center is nonhyperbolic, it is not structurally stable in general



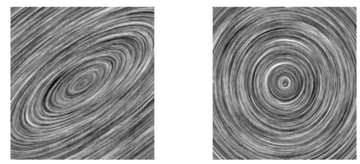
but structurally stable if the field is divergence-free.

Center

- zero trace
- positive determinant
- negative discriminant

counter-clockwise if $\partial v/\partial x - \partial u/\partial y > 0$

Example



$$\mathbf{J} = \begin{pmatrix} 0.98 & -1.885 \\ 1.04 & -0.98 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{A}$$

Other stationary points

Other stationary points in 2D:

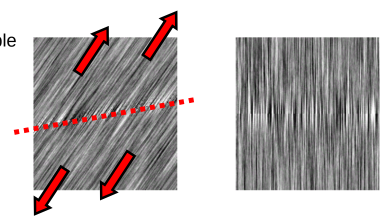
If **J** is a singular matrix, the following stationary (but not critical!) points are possible:

- if a single eigenvalue is zero: line source, line sink
- if both eigenvalues are zero : pure shear

Line source

- positive trace
- zero determinant

Example

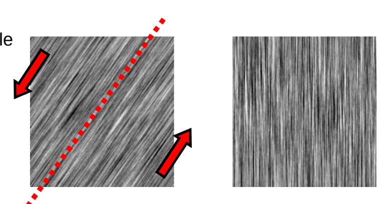


$$\mathbf{J} = \begin{pmatrix} -0.15 & 0.8625 \\ -0.2 & 1.15 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{A}$$

Pure shear

- zero trace
- zero determinant

Example



$$\mathbf{J} = \begin{pmatrix} 0.75 & -0.5625 \\ 1 & -0.75 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{A}$$

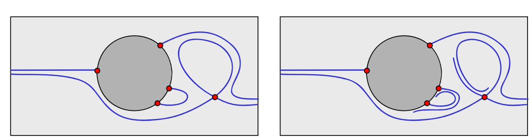
The topological skeleton

The topological skeleton consists of all periodic orbits and all streamlines converging (in either direction of time) to

- a saddle point (separatrix of the saddle), or
- a critical point on a no-slip boundary

It provides a kind of segmentation of the 2D vector field

Examples:



Example: irrotational vector fields.

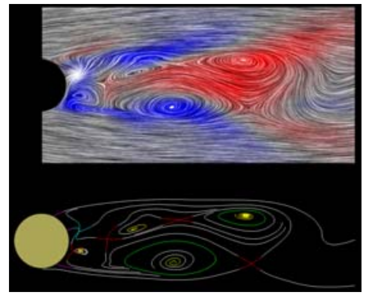
An irrotational (conservative) vector field is the gradient of a scalar field (its potential).

Skeleton of an irrotational vector field: watershed image of its potential field.

Discussion:

- watersheds are topologically defined, integration required
- height ridges are geometrically defined, locally detectable

Example: LIC and topology-based visualization (skeleton plus a few extra streamlines).



Example: topological skeleton of a surface flow

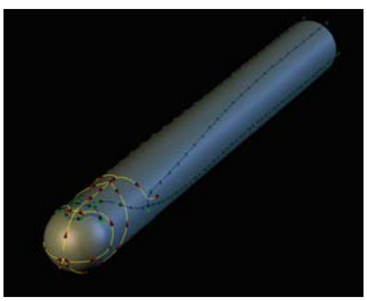


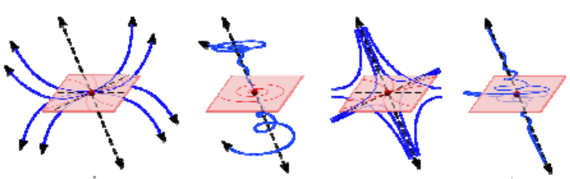
image credit: A. Globus

Critical points in 3D

Hyperbolic critical points in 3D can be classified as follows:

- three real eigenvalues:
  - all positive: source
  - two positive, one negative: 1:2 saddle (1 in, 2 out)
  - one positive, two negative: 2:1 saddle (2 in, 1 out)
  - all negative: sink
- one real, two complex eigenvalues:
  - positive real eigenvalue, positive real parts: spiral source
  - positive real eigenvalue, negative real parts : 2:1 spiral saddle
  - negative real eigenvalue, positive real parts : 1:2 spiral saddle
  - negative real eigenvalue, negative real parts : spiral sink

Types of hyperbolic critical points in 3D



source spiral source 2:1 saddle 2:1 spiral saddle

The other 4 types are obtained by reversing arrows

Example: The Lorenz attractor

The Lorenz attractor

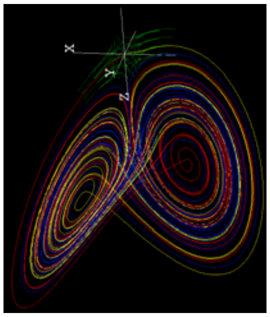
$$\mathbf{v} = \left(10(y-x),\ 28x - y - xz,\ xy - 8z/3\right)$$

has 3 critical points:

- a 2:1 saddle P_0
  - at $(0,0,0)$
  - with eigenvalues $\{-22.83, -2.67, 11.82\}$
- two 1:2 spiral saddles P_1 and P_2
  - at $\left(-6\sqrt{2}, -6\sqrt{2}, 27\right)$ and $\left(6\sqrt{2}, 6\sqrt{2}, 27\right)$
  - with eigenvalues $\{-13.85, 0.09 \pm 10.19i\}$

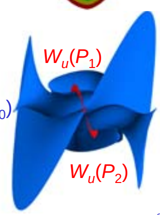
Example: The Lorenz attractor

Streamlines



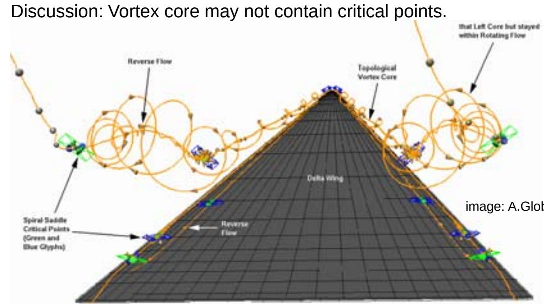
Streamsurfaces (2D separatrices)

stable manifold $W_s(P_0)$



Visualization based on 3D critical points

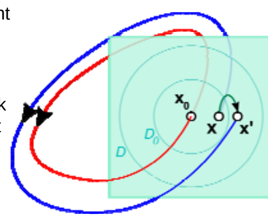
Example: Flow over delta wing, glyphs (icons) for critical point types, 1D separatrices ("topological vortex cores").
Discussion: Vortex core may not contain critical points.



Periodic orbits

Poincaré map of a periodic orbit in 3D:

- Choose a point $\mathbf{x}_0$ on the periodic orbit
- Choose an open circular disk D centered at $\mathbf{x}_0$
  - on a plane which is not tangential to the flow, and
  - small enough that the periodic orbit intersects D only in $\mathbf{x}_0$
- Any streamline seeded at a point $\mathbf{x} \in D$ which intersects D a next time at a point $\mathbf{x}' \in D$ defines a mapping from $\mathbf{x}$ to $\mathbf{x}'$
- There exists a smaller open disk $D_0 \subseteq D$ centered at $\mathbf{x}_0$ such that this mapping is defined for all points $\mathbf{x} \in D_0$.
- This is the Poincaré map.



Periodic orbits

Using coordinates on the plane of D and with origin at $\mathbf{x_0}$, the Poincaré map can now be linearized:

$$\mathbf{x} \mapsto \mathbf{Px}$$

where $\mathbf{P}$ is 2x2 matrix.

Important fact about Poincaré maps:
The eigenvalues of $\mathbf{P}$ are independent of

- the choice of $\mathbf{x_0}$ on the periodic orbit
- the orientation of the plane of D
- the choice of coordinates for the plane

A periodic orbit is called hyperbolic, if its eigenvalues lie off the complex unit circle. Hyperbolic p.o. are structurally stable.

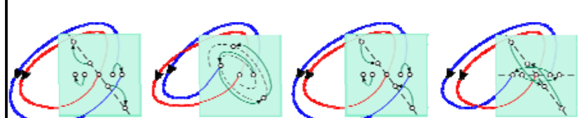
Periodic orbits

Hyperbolic periodic orbits in 3D can be classified as follows:

- Two real eigenvalues:
  - both outside the unit circle: source p.o.
  - both inside the unit circle: sink p.o.
  - one outside, one inside:
    - both positive: saddle p.o.
    - both negative: twisted saddle p.o.
- Two complex conjugate eigenvalues:
  - both outside the unit circle: spiral source p.o.
  - both inside the unit circle: spiral sink p.o.

Periodic orbits

Types of hyperbolic periodic orbits in 3D



source p.o. spiral source p.o. saddle p.o. twisted saddle p.o.

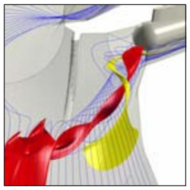
Types sink and spiral sink are obtained by reversing arrows.

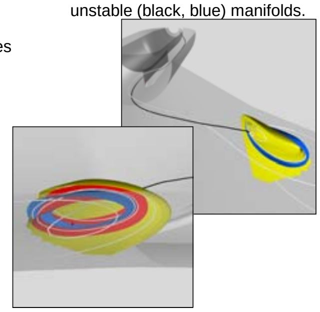
Periodic orbits

Example: Flow in Pelton distributor ring.

Critical point of spiral saddle type and p.o. of twisted saddle type. Stable (yellow, red) and unstable (black, blue) manifolds.

Streamlines and streamsurfaces (manually seeded).





Saddle connectors

The topological skeleton of 3D vector fields contains 1D and 2D separatrices of (spiral) saddles.

Not directly usable for visualization (too much occlusion).

Alternative: only show intersection curves of 2D separatrices.

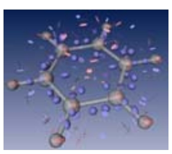
Two types of saddle connectors:

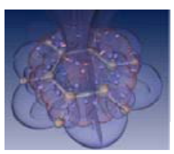
- heteroclinic orbit: connects two (spiral) saddles
- homoclinic orbits: connects a (spiral) saddle with itself

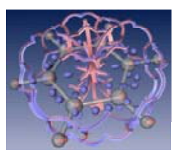
Idea: a 1D "skeleton" is obtained, not providing a segmentation, but indicating flow between pairs of saddles

Saddle connectors

Comparison: icons / full topological skeleton / saddle connectors







Flow past a cylinder:

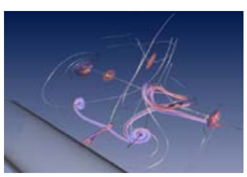
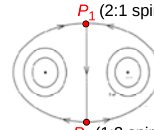


Image credit: H. Theisel

Saddle connectors

In rotational flow, a connected pair of spiral saddles can describe a vortex breakdown bubble.

- ideal case:
  - $W_s(P_1)$ coincides with $W_u(P_2)$
  - no saddle connector
- perturbed case:
  - transversal intersection of $W_s(P_1)$ and $W_u(P_2)$
  - saddle connector consists of two streamlines



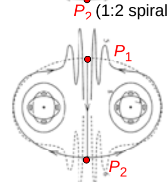


Image credit: Krasny/Nitsche

Saddle connectors

3D view

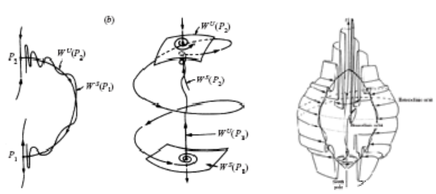


Image credit: Sotiropoulos et al.

If **v** is velocity field of a fluid:

- Folds must have constant mass flux.
- Close to P_1 or P_2 this is approximately $\int \rho \mathbf{v} \cdot d\mathbf{n} \approx \rho \omega A r$ (density * angular velocity * cross section area * radius).
- It follows: cross section area ~ 1/radius
- Consequence: Shilnikov chaos

Saddle connectors

- Experimental photograph of a vortex breakdown bubble
- Vortex breakdown bubble in flow over delta wing, visualization by streamsurfaces (not topology-based)

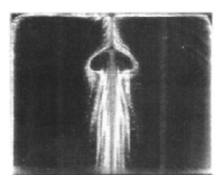
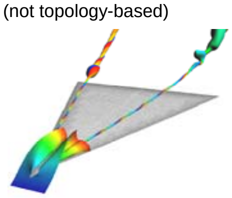


Image credit: Sotiropoulos et al.



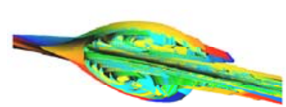


Image credit: C. Garth

Saddle connectors

- Vortex breakdown bubble found in CFD data of Francis draft tube: