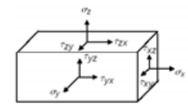
9

Tensor Field Visualization

Tensors

"Tensors are the language of mechanics"

Tensor of order (rank)
- 0: scalar
- 1: vector
- 2: matrix
- …



(example: stress tensor)

Tensors can have "lower" and "upper" indices, e.g. a_{ij}, a_i^j, a^{ij}, indicating different transformation rules for change of coordinates.

Tensors

Visualization methods for tensor fields:
- tensor glyphs
- tensor field lines, hyperstreamlines
- tensor field topology
- fiber bundle tracking

Tensor field visualization only deals with 2nd order tensors (matrices).

→ eigenvectors and eigenvalues contain full information.

Separate visualization methods for symmetric and nonsymmetric tensors.

Tensor glyphs

In 3D, tensors are 3x3 matrices.

The velocity gradient tensor is nonsymmetric → 9 degrees of freedom for the local change of the velocity vector.

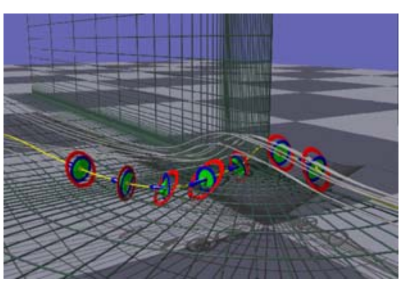
A glyph developed by de Leeuw and van Wijk can visualize all these 9 DOFs:

- tangential acceleration (1): green "membrane"
- orthogonal acceleration (2): curvature of arrow
- twist (1): candy stripes
- shear (2): orange ellipse (gray ellipse for ref.)
- convergence/divergence (3): white "parabolic reflector"

Tensor glyphs

Example:

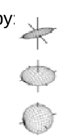
NASA "bluntfin" dataset, glyphs shown on points on a streamline.



Tensor glyphs

Symmetric 3D tensors have real eigenvalues and orthogonal eigenvectors → they can be represented by ellipsoids.

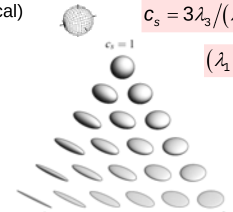
Three types of anisotropy:
- linear anisotropy
- planar anisotropy
- isotropy (spherical)



Anisotropy measure:

$$c_l = (\lambda_1 - \lambda_2)/(\lambda_1 + \lambda_2 + \lambda_3)$$
$$c_p = 2(\lambda_2 - \lambda_3)/(\lambda_1 + \lambda_2 + \lambda_3)$$
$$c_s = 3\lambda_3/(\lambda_1 + \lambda_2 + \lambda_3)$$

$$(\lambda_1 \geq \lambda_2 \geq \lambda_3)$$

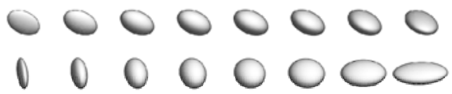


Images: G. Kindlmann

Problem of ellipsoid glyphs:

- shape is poorly recognized in projected view

Example: 8 ellipsoids, 2 views

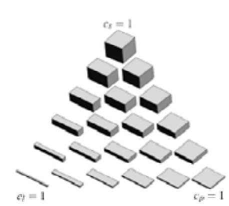


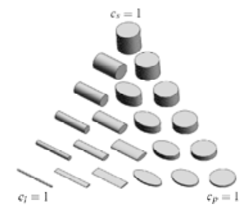
Problem of cuboid glyphs:

- small differences in eigenvalues are over-emphasized

Problems of cylinder glyphs:

- discontinuity at $c_l = c_p$
- artificial orientation at $c_s = 1$





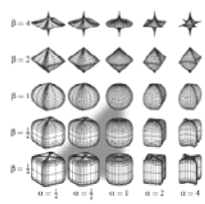
Combining advantages: superquadrics

Superquadrics with *z* as primary axis:

$$\mathbf{q}_z(\theta,\phi) = \begin{pmatrix} \cos^\alpha\theta\sin^\beta\phi \\ \sin^\alpha\theta\sin^\beta\phi \\ \cos^\beta\phi \end{pmatrix}$$

$$0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$$

with $\cos^\alpha\theta$ used as shorthand for $|\cos\theta|^\alpha \operatorname{sgn}(\cos\theta)$



Superquadrics for some pairs (α,β)

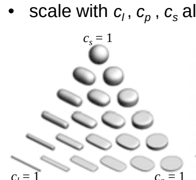
Shaded: subrange used for glyphs

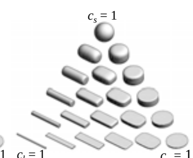
Superquadric glyphs (Kindlmann): Given c_l, c_p, c_s

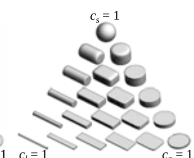
- compute a base superquadric using a sharpness value γ:

$$q(\theta,\phi) = \begin{cases} \text{if } c_l \ge c_p:\ q_z(\theta,\phi) \text{ with } \alpha = (1-c_p)^\gamma \text{ and } \beta = (1-c_l)^\gamma \\ \text{if } c_l < c_p:\ q_x(\theta,\phi) \text{ with } \alpha = (1-c_l)^\gamma \text{ and } \beta = (1-c_p)^\gamma \end{cases}$$

- scale with c_l, c_p, c_s along *x,y,z* and rotate into eigenvector frame

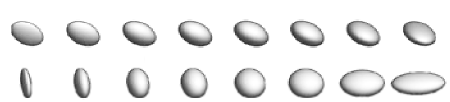






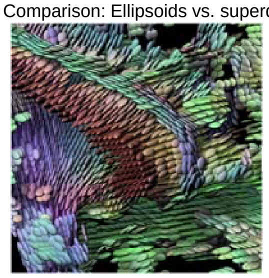
Comparison of shape perception (previous example)

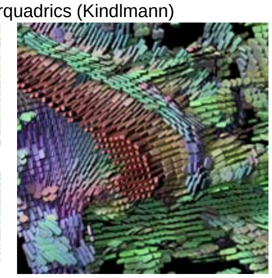
- with ellipsoid glyphs



- with superquadrics glyphs

Comparison: Ellipsoids vs. superquadrics (Kindlmann)





color map: $\begin{pmatrix} R \\ G \\ B \end{pmatrix} = c_l \begin{pmatrix} |e_x^1| \\ |e_y^1| \\ |e_z^1| \end{pmatrix} + (1-c_l)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (with e^1 = major eigenvector)

Tensor field lines

Let $\mathbf{T}(\mathbf{x})$ be a (2nd order) symmetric tensor field.

$\rightarrow$ real eigenvalues, orthogonal eigenvectors

Tensor field line: by integrating along one of the eigenvectors

Important: Eigenvector fields are not vector fields!

- eigenvectors have no magnitude and no orientation (are bidirectional)
- the choice of the eigenvector can be made consistently as long as eigenvalues are all different
- tensor field lines can intersect only at points where two or more eigenvalues are equal, so-called degenerate points.

Tensor field lines

Tensor field lines can be rendered as hyperstreamlines:
tubes with elliptic cross section, radii proportional to 2nd and 3rd eigenvalue.

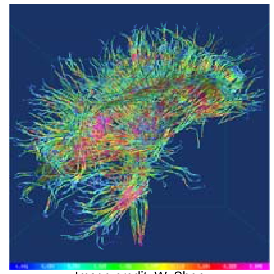


Image credit: W. Shen

Tensor field topology

Based on tensor field lines, a tensor field topology can be defined, in analogy to vector field topology.

Degenerate points play the role of critical points:

At degenerate points, infinitely many directions (of eigenvectors) exist.

For simplicity, we only study the 2D case.

For locating degenerate points: solve equations

$$T_{11}(\mathbf{x}) - T_{22}(\mathbf{x}) = 0, \quad T_{12}(\mathbf{x}) = 0$$

Tensor field topology

It can be shown:

The type of the degenerated point depends on

$$\delta = ad - bc$$

where

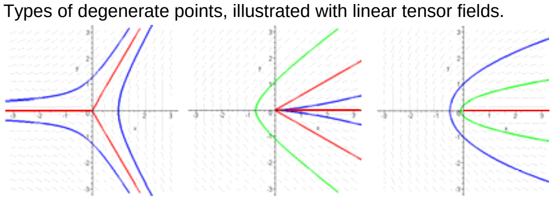
$$a = \frac{1}{2}\frac{\partial(T_{11} - T_{22})}{\partial x} \qquad b = \frac{1}{2}\frac{\partial(T_{11} - T_{22})}{\partial y}$$

$$c = \frac{\partial T_{12}}{\partial x} \qquad d = \frac{\partial T_{12}}{\partial y}$$

- for $\delta<0$ the type is a trisector
- for $\delta>0$ the type is a wedge
- for $\delta=0$ the type is structurally unstable

Tensor field topoloy

Types of degenerate points, illustrated with linear tensor fields.



trisector

$$\mathbf{T} = \begin{pmatrix} 1-2x & y \\ y & 1 \end{pmatrix}$$

$$\mathbf{e} = \begin{pmatrix} \sqrt{x^2+y^2} - x \\ y \end{pmatrix}$$

$$\delta = -1$$

double wedge

$$\mathbf{T} = \begin{pmatrix} 1+2x/3 & y \\ y & 1 \end{pmatrix}$$

$$\mathbf{e} = \begin{pmatrix} x + \sqrt{x^2+9y^2} \\ 3y \end{pmatrix}$$

$$\delta = 1/3$$

single wedge

$$\mathbf{T} = \begin{pmatrix} 1+x & y \\ y & 1-x \end{pmatrix}$$

$$\mathbf{e} = \begin{pmatrix} y \\ \sqrt{x^2+y^2} - x \end{pmatrix}$$

$$\delta = 1$$

Tensor field topology

Separatrices are tensor field lines converging to the degenerate point with a radial tangent.

They are straight lines in the special case of a linear tensor field.

Double wedges have one "hidden separatrix" and two other separatrices which actually separate regions of different field line behavior.

Single wedges have just one separatrix.

The angles of the separatrices are obtained by solving:

$$dm^3 + (c + 2b)m^2 + (2a - d)m - c = 0$$

If $m \in \mathbb{R}$, the two angles

$$\theta = \pm \arctan m$$

are angles of a separatrix. The two choices of signs correspond to the two choices of tensor field lines (minor and major eigenvalue).

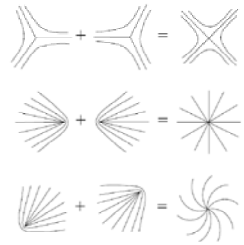
If $d = 0$, an additional solution is

$$\theta = \pm 90°$$

There are in general 1 or 3 real solutions:

- 3 separatrices for trisector and double wedge
- 1 separatrix for single wedge

Saddles, nodes, and foci can exists as nonelementary (higher-order) degenerate points with δ=0. They are created by merging trisectors or wedges. They are not structurally stable and break up in their elements if perturbed.



The topological skeleton is defined as the set of separatrices of trisector points.

Example: Topological transition of the stress tensor field of a flow past a cylinder

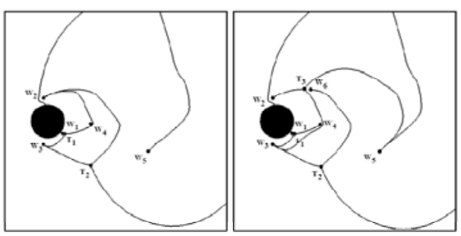


Image credit: T. Delmarcelle

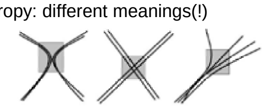
Diffusion tensor imaging (DTI) is a newer magnetic resonance imaging (MRI) technique.

DTI produces a tensor field of the anisotropy of the brain's white matter.

Most important application: Tracking of fiber bundles.

Interpretation of anisotropy types:

- isotropy: no white matter
- linear anisotropy: direction of fiber bundle
- planar anisotropy: different meanings(!)



Fiber bundle tracking ≠ tensor field line integration, because bundles may cross each other

Method 1:

Best neighbor algorithm (Poupon), based on idea of restricting the curvature:

- at each voxel compute eigenvector of dominant eigenvalue
  → "direction map"
- at each voxel *M* find "best neighbor voxel" *P* according to angle criterion (mimimize max of $\alpha_1,\alpha_2,\alpha_3$ over 26 neighbors)
  → "tracking map"
- connect voxels (within a "white matter mask") with its best neighbor.

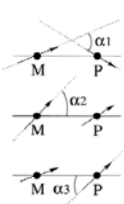


Image credit: C. Poupon

Method 2:

Apply moving least squares filter which favors current direction of the fiber bundle (Zhukov and Barr).

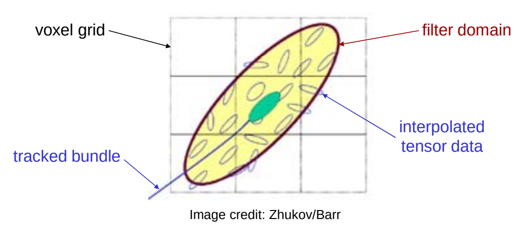


Image credit: Zhukov/Barr

DTI fiber bundle tracking

Method 3:
Tensor deflection (TEND) method (Lazar et al.)

Idea: if **v** is the incoming bundle direction, use **Tv** as the direction of the next step.

Reasoning:

- **Tv** bends the curve towards the dominant eigenvector
- **Tv** has the unchanged direction of **v** if **v** is an eigenvector of **T** or a vector within the eigenvector plane if the two dominant eigenvalues are equal (rotationally symmetric **T**).

DTI fiber bundle tracking

Comparison:
Tensor field lines (l), TEND (m), weighted sum (r),
Stopping criteria: fractional anisotropy < 0.15 or angle between successive steps > 45 degrees

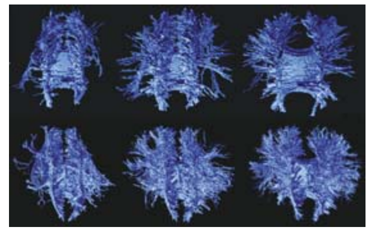
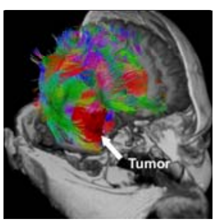


image credit: M. Lazar

DTI fiber bundle tracking

Clustering of fibers: Goal is to identify nerve tracts.

automatic clustering results



optic tract (orange) and pyramidal tract (blue).

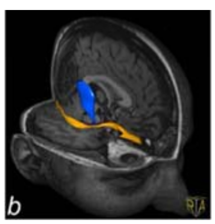


image credit: Merhof et al. / Enders et al.

DTI fiber bundle tracking

Algorithmic steps

1. clustering based on geometric attributes: centroid, variance, curvature, …
2. center line: find sets of "matching vertices" and average them
3. wrapping surface: compute convex hull in orthogonal slices, using Graham's Scan algorithm