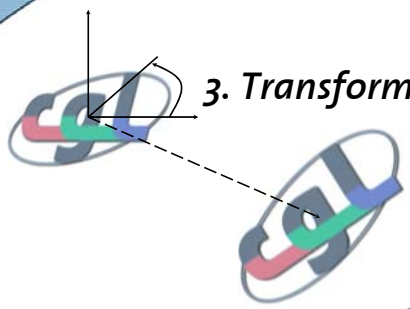


3. Transformations

Linear and Affine Mappings

- a map A: $\boldsymbol{x} \Rightarrow \boldsymbol{x'}$ is called ***linear***, if

$$A(\alpha \boldsymbol{x} + \beta \boldsymbol{y}) = \alpha A(\boldsymbol{x}) + \beta A(\boldsymbol{y})$$

- a map A: $\boldsymbol{x} \Rightarrow \boldsymbol{x'}$ is called ***affine***, if

$$\boldsymbol{x'} = A(\boldsymbol{x}) + \boldsymbol{t} = B(\boldsymbol{x})$$

- Linear transforms are represented by matrices, i.e. $\boldsymbol{Ax} = \boldsymbol{x'}$

2D Transforms

- Translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} dx \\ dy \end{pmatrix} \Rightarrow \boldsymbol{P'} = \boldsymbol{P} + \boldsymbol{T}$$

- Scaling

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \boldsymbol{P'} = \boldsymbol{S} \times \boldsymbol{P}$$

2D Transforms

- Rotation of a point $\boldsymbol{P}$ along some angle θ

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$

- Matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \boldsymbol{P'} = \boldsymbol{R} \times \boldsymbol{P}$$

Homogenous Coordinates

- Matrix form of 3 fundamental transforms

$$\boldsymbol{P'} = \boldsymbol{R} \times \boldsymbol{P} \qquad \boldsymbol{P'} = \boldsymbol{S} \times \boldsymbol{P} \qquad \boldsymbol{P'} = \boldsymbol{T} + \boldsymbol{P}$$

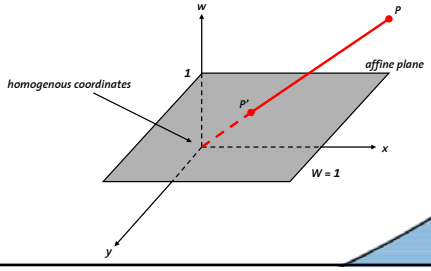
- Translation as additive component
- Take point $\boldsymbol{P}$ in homogenous coordinates by adding additional weight:

$$\boldsymbol{P} = (x, y, W)$$

A point P in 2D has infinitely many homogenous coordinates: (2,1) can be written as (2,1,1), (4,2,2) or (-4,-2,-2)... Division by w -> projection

Homogenous Coordinates

- Point $\boldsymbol{P'} = (x, y, 1)$ as line (wx, wy, w) in 3D



Translation and Scaling

- Representation by 3x3 matrix

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \times \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \Rightarrow \boldsymbol{P'} = \boldsymbol{T} \times \boldsymbol{P}$$

- Scaling

$$\boldsymbol{S}(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation and Shearing

- Rotation Matrix

$$\boldsymbol{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Shears along *x*- and *y*-axis

$$\boldsymbol{SH}_x = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{SH}_y = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Combinations

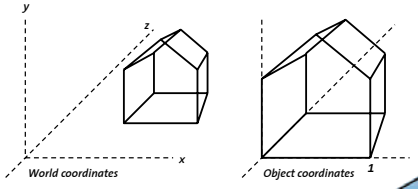
- Stack transforms using matrix multiplication
- Example: Translation and Rotation $\boldsymbol{T} \times \boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$
- Commutativity of Matrices ***M1***, ***M2*** given if

Matrix M1	*Matrix M2*
Translation	*Translation*
Rotation	*Rotation*
Scaling	*Scaling*
Scaling	*Rotation*

2D only!

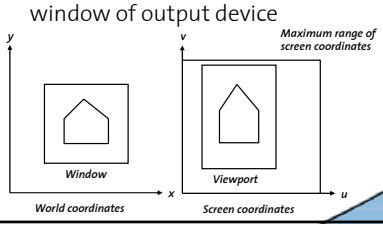
Coordinate Systems

- ***World coordinates***: reference system to describe scenery
- ***Object coordinates***: internal representation

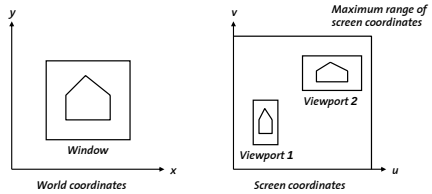


Windows and Viewports

- ***Window*** in world coordinates to represent camera
- ***Viewport*** in device coordinates to define sub-window of output device

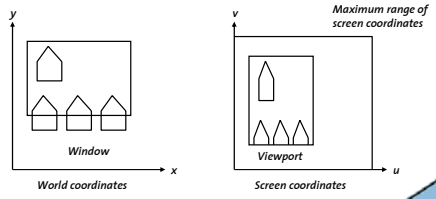


Multiple Viewports



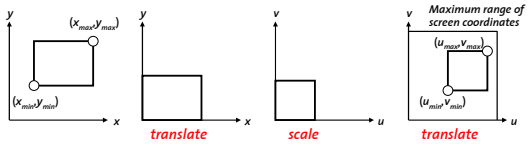
Clipping Windows

- Transform affects clipping and ***aspect ratio***



Window-Viewport Transform

- 3 step sequence of translate-scale-translate



$$M_{wv} = T(u_{min}, v_{min}) \times S\left(\frac{u_{max}-u_{min}}{x_{max}-x_{min}}, \frac{v_{max}-v_{min}}{y_{max}-y_{min}}\right) \times T(-x_{min}, -y_{min})$$

Window-Viewport Transform

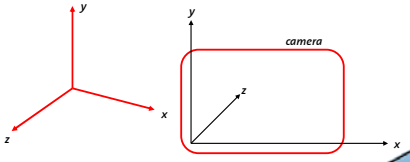
- Homogenous coordinates

$$= \begin{bmatrix} 1 & 0 & u_{min} \\ 0 & 1 & v_{min} \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \frac{u_{max}-u_{min}}{x_{max}-x_{min}} & 0 & 0 \\ 0 & \frac{v_{max}-v_{min}}{y_{max}-y_{min}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -x_{min} \\ 0 & 1 & -y_{min} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{u_{max}-u_{min}}{x_{max}-x_{min}} & 0 & -x_{min} \cdot \frac{u_{max}-u_{min}}{x_{max}-x_{min}} + u_{min} \\ 0 & \frac{v_{max}-v_{min}}{y_{max}-y_{min}} & -y_{min} \cdot \frac{v_{max}-v_{min}}{y_{max}-y_{min}} + v_{min} \\ 0 & 0 & 1 \end{bmatrix}$$

3D Transforms

- Homogenous coordinates -> 4x4 matrices
- Project point $\boldsymbol{p} = (x, y, z, w)$ onto hyperplane $(x/w, y/w, z/w, 1)$
- Left-handed versus right-handed systems



Translation and Scaling

- Representation by 4x4 matrix

$$T(t_x, t_y, t_z) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Scaling

$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation

- Rotation matrices for x-,y-, and z-axis

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about Arbitrary Axis

- Normalized axis **u**, angle θ

$$\mathbf{R}(\mathbf{u},\theta) = \begin{bmatrix} u_x^2+\cos\theta(1-u_x^2) & u_x u_y(1-\cos\theta)-u_z\sin\theta & u_x u_z(1-\cos\theta)+u_y\sin\theta & 0 \\ u_x u_y(1-\cos\theta)+u_z\sin\theta & u_y^2+\cos\theta(1-u_y^2) & u_y u_z(1-\cos\theta)+u_x\sin\theta & 0 \\ u_x u_z(1-\cos\theta)-u_y\sin\theta & u_y u_z(1-\cos\theta)+u_x\sin\theta & u_z^2+\cos\theta(1-u_z^2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shear

- Shearing parallel to principal planes

$$SH_{xy}(sh_x, sh_y) = \begin{bmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad SH_{xz}(sh_x, sh_z) = \begin{bmatrix} 1 & sh_x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & sh_z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$SH_{yz}(sh_y, sh_z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ sh_y & 1 & 0 & 0 \\ sh_z & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transform of Normal Vector

- Given a plane in implicit form

$$Ax + By + Cz + D = 0$$

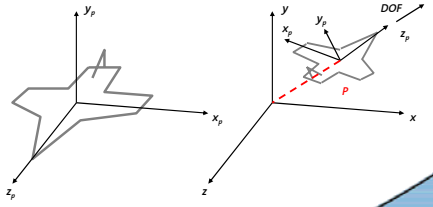
- Normal $\boldsymbol{n} = (A, B, C, D)$ and $\boldsymbol{P} = (x, y, z, 1)$
- Transformed normal $\boldsymbol{n'}$ computed by

$$\boldsymbol{n}' = (\boldsymbol{M}^{-1})^T \boldsymbol{n}$$

Verification by insertion into plane equation and some algebra !

Compound 3D Transforms

- Object-to-world coordinate transform
- Example: Visual simulation



Compound 3D Transforms

1. Compute a new orthogonal basis by
   - $\mathbf{r}_1 = \boldsymbol{y} \times \boldsymbol{DOF}$
   - $\mathbf{r}_2 = \boldsymbol{DOF} \times (\boldsymbol{y} \times \boldsymbol{DOF})$ (orthogonal)
   - $\mathbf{r}_3 = \boldsymbol{DOF}$
2. Corresponding rotation matrix is

$$\mathbf{R} = \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} & 0 \\ r_{1y} & r_{2y} & r_{3y} & 0 \\ r_{1z} & r_{2z} & r_{3z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Compound 3D Transforms

2. Translate into $\boldsymbol{P} = (p_x, p_y, p_z, 1)$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

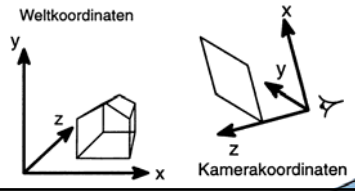
3. Compound model matrix

$$\mathbf{M} = \mathbf{TR} = \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} & p_x \\ r_{1y} & r_{2y} & r_{3y} & p_y \\ r_{1z} & r_{2z} & r_{3z} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Model-View Transform

Model and View transforms are inverse to each other !

1. Left-Right transform
2. Rotate into new basis (camera)
3. Translate
4. Invert compound matrix



Modeling Transform – Forward

$$M_T \times M_R \times M_{LR} = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_{x_1} & h_{y_1} & h_{z_1} & 0 \\ h_{x_2} & h_{y_2} & h_{z_2} & 0 \\ h_{x_3} & h_{y_3} & h_{z_3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} h_{x_1} & h_{y_1} & -h_{z_1} & P_x \\ h_{x_2} & h_{y_2} & -h_{z_2} & P_y \\ h_{x_3} & h_{y_3} & -h_{z_3} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Viewing Transform – Inverse

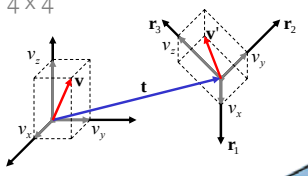
$$M_{LR}^{-1} \times M_R^{-1} \times M_T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_{x_1} & h_{x_2} & h_{x_3} & 0 \\ h_{y_1} & h_{y_2} & h_{y_3} & 0 \\ h_{z_1} & h_{z_2} & h_{z_3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -P_x \\ 0 & 1 & 0 & -P_y \\ 0 & 0 & 1 & -P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note the orthogonality of M_R-> transpose is inverse !

Duality of Modeling Transform and Viewing Transform (OpenGL)

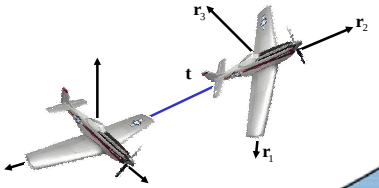
Transformation between Coordinate Systems

$$\underbrace{\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{4 \times 4} \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{t} + v_x\mathbf{r}_1 + v_y\mathbf{r}_2 + v_z\mathbf{r}_3 \\ 1 \end{bmatrix}$$



Model to World

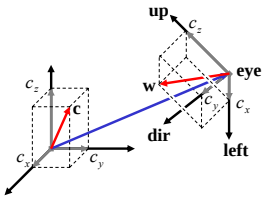
$$\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} m_x \\ m_y \\ m_z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{t} + v_x\mathbf{r}_1 + v_y\mathbf{r}_2 + v_z\mathbf{r}_3 \\ 1 \end{bmatrix} = \begin{bmatrix} w_x \\ w_y \\ w_z \\ 1 \end{bmatrix}$$



World to Camera

$$\begin{bmatrix} \mathbf{dir} & \mathbf{up} & -\mathbf{left} & \mathbf{eye} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_x \\ c_y \\ c_z \\ 1 \end{bmatrix} = \begin{bmatrix} w_x \\ w_y \\ w_z \\ 1 \end{bmatrix}$$

- Solve for **c**
- Invert Transformation



An OpenGL Example

- View Transform for entire scene

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
gluLookAt(0,0,2, 0,0,0, 0,1,0);  // eye, center, up
```

- Model Transform only for teapot

```
glMatrixMode(GL_MODELVIEW);
glPushMatrix();
glRotatef(30, 1.0, 1.0, 1.0);
glutSolidTeapot(0.5);
glPopMatrix();
```


Quaternions

- Elegant notion for the modeling of translation and rotation
- Compact representation
- Mathematical element with a set of operators (e.g. multiplication)
- Efficient implementation
- Widely used for animation

Definition

- A ***quaternion*** $\boldsymbol{q}$ is a "hypercomplex" number defined as: $\boldsymbol{q} = c + x\,i + y\,j + z\,k$
- c, y, x, z are real numbers
- i, j, k are imaginary
- Notation: $\boldsymbol{q} = c + \boldsymbol{u}$
- c: real part, $\boldsymbol{u}$: ***pure quaternion***
  $\boldsymbol{u} = x\,i + y\,j + z\,k$

Basic Properties and Operators

- Addition
  $\boldsymbol{q} + \boldsymbol{q}' = (c + c') + (x + x')\,i + (y + y')\,j + (z + z')\,k$
- Multiplication of complex operators
  $i^2 = j^2 = k^2 = -1$
  $i\,j = k,\quad j\,i = -k;\quad j\,k = i,\quad k\,j = -i;\quad k\,i = j,\quad i\,k = -j$
- Quaternion multiplication of $\boldsymbol{q}$ and $\boldsymbol{q}'$
  $\boldsymbol{q}\boldsymbol{q}' = (c + u)(c' + u')$
  $= (c\,c' - \boldsymbol{u}\cdot\boldsymbol{u}') + (\boldsymbol{u}\times\boldsymbol{u}' + < c\,\boldsymbol{u}' > + < c'\,\boldsymbol{u} >)$

Quaternion multiplication is NOT commutative!

Basic Properties and Operators

- Inner product, scalar operator **<...>** and cross product **×** defined as:
  $\boldsymbol{u}\cdot\boldsymbol{u}' = x\,x' + y\,y' + z\,z'$
  $< c\,\boldsymbol{u} > = c\,x\,i + c\,y\,j + c\,z\,k$
  $\boldsymbol{u}\times\boldsymbol{u}' = (y\,z' - z\,y')\,i + (z\,x' - x\,z')\,j + (x\,y' - y\,x')\,k$
- One-elements of addition and multiplication
  $0 = 0 + 0\,i + 0\,j + 0\,k$
  $1 = 1 + 0\,i + 0\,j + 0\,k$

Basic Properties and Operators

- Conjugate elements
  $\boldsymbol{q} = c + \boldsymbol{u}\quad ;\quad \bar{\boldsymbol{q}} = c - \boldsymbol{u}$
- Absolute
  $\boldsymbol{q}\bar{\boldsymbol{q}} = c^2 + x^2 + y^2 + z^2$
  $\boldsymbol{q}\bar{\boldsymbol{q}} = |\boldsymbol{q}|^2$
- Inverse elements of
  - addition $-\boldsymbol{q} = -c - x\,i - y\,j - z\,k$
  - multiplication $\boldsymbol{q}^{-1} = \frac{1}{|\boldsymbol{q}|^2}\bar{\boldsymbol{q}}$

Unit Quaternions

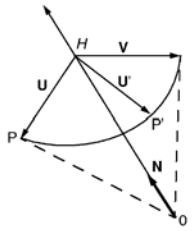
- Quaternions of length 1 are fundamental to encode transformations
- Using a unit quaternion.. $|\boldsymbol{q}|^2 = c^2 + \boldsymbol{u}\cdot\boldsymbol{u} = 1$
- .. and introducing .. $\boldsymbol{N} = [N_x, N_y, N_z]^T \quad \boldsymbol{I} = [i, j, k]^T$
- ..with.. $\boldsymbol{q} = c + \boldsymbol{u} \quad \boldsymbol{u} = s\boldsymbol{n} \quad \boldsymbol{n} = \boldsymbol{NI}$
- ..it follows that $c^2 + s^2 = 1$
- Each unit quaternion can be rewritten as

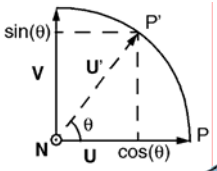
$$\boldsymbol{q} = \cos\theta + \sin\theta\,\boldsymbol{n}$$

$$\Rightarrow \boldsymbol{q}\boldsymbol{q}' = \cos(\theta+\theta') + \sin(\theta+\theta')\boldsymbol{n}$$

3D Rotation using Quaternions

- Rotation of a point **P** along an arbitrary axis **N** and angle θ





3D Rotation using Quaternions

- Find **P'** as a function of **P** and **N**

$$\boldsymbol{P}' = \cos\theta\,\boldsymbol{P} + (1-\cos\theta)\boldsymbol{N}(\boldsymbol{N}\cdot\boldsymbol{P}) + \sin\theta(\boldsymbol{N}\times\boldsymbol{P})$$

- Corresponding rotation operator $\boldsymbol{R}(\theta, \boldsymbol{N})$

$$\boldsymbol{R}(\theta,\boldsymbol{N}) = \cos\theta\,\boldsymbol{I}_3 + (1-\cos\theta)\boldsymbol{N}^T\boldsymbol{N} + \sin\theta\,\boldsymbol{A}_N$$

$$\boldsymbol{N} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}, \quad \boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}_N = \begin{bmatrix} 0 & N_3 & -N_2 \\ -N_3 & 0 & N_1 \\ N_2 & -N_1 & 0 \end{bmatrix}$$

3D Rotation using Quaternions

- Represent point **P** as a pure quaternion

$$\boldsymbol{P} = (x, y, z) \qquad \boldsymbol{p} = 0 + \boldsymbol{v} = xi + yj + zk$$

- Rotation using quaternion operators

$$\boldsymbol{R}_q(\boldsymbol{p}) = \boldsymbol{q}\,\boldsymbol{p}\,\bar{\boldsymbol{q}} \qquad \boldsymbol{q} = c + \boldsymbol{u} = \cos\theta + \sin\theta\,\boldsymbol{n}$$

- Insertion and a little algebra reveals ..

$$\boldsymbol{R}_q(\boldsymbol{p}) = \begin{bmatrix} <\cos(2\theta)\boldsymbol{v}> + \\ <(1-\cos(2\theta))(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} + <\sin(2\theta)(\boldsymbol{n}\times\boldsymbol{v})> \end{bmatrix}$$

- Rotation of **P** along axis **N** by angle 2θ

Points and Rotation

- Recipe: Take **p** and **q** and compute

$$\boldsymbol{p}' = \boldsymbol{q}\,\boldsymbol{p}\,\bar{\boldsymbol{q}}$$

$$\boldsymbol{q} = \cos(\theta/2) + \sin(\theta/2)\boldsymbol{n}$$

$$\boldsymbol{n} = N_1 i + N_2 j + N_3 k$$

- Elegant implementation using operator overloading

Points and Translation

- Translation vector as a pure quaternion

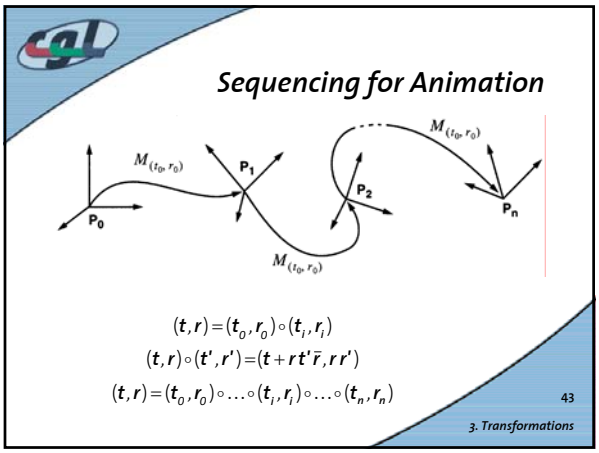
$$\boldsymbol{p}' = \boldsymbol{p} + \boldsymbol{t}$$

- Sequences of rotations **r** and translations **t** using a transformation operator **M**

$$\boldsymbol{p} \rightarrow \boldsymbol{p}' = \boldsymbol{M}_{(t,r)}(\boldsymbol{p}) = \boldsymbol{r}\,\boldsymbol{p}\,\bar{\boldsymbol{r}} + \boldsymbol{t}$$

- $\boldsymbol{M}_{(t,r)}$ denotes rotation-translation-operator
- $\boldsymbol{M}_{(0,r)}$ describes rotation and $\boldsymbol{M}_{(t,1)}$ translation

$$\boldsymbol{M}_{(t,r)} = \boldsymbol{M}_{(0,r)} \circ \boldsymbol{M}_{(t,1)}$$


Sequencing for Animation
$M_{(t_0, r_0)}$
P_1
P_0
P_2
$M_{(t_0, r_0)}$
$M_{(t_0, r_0)}$
P_n
$M_{(t_0, r_0)}$
$(t, r) = (t_0, r_0) \circ (t_i, r_i)$
$(t, r) \circ (t', r') = (t + r t' \bar{r}, r r')$
$(t, r) = (t_0, r_0) \circ \ldots \circ (t_i, r_i) \circ \ldots \circ (t_n, r_n)$
43
3. Transformations