T5 – Variational Calculus

Christian Vögeli
cvoegeli@inf.ethz.ch

Motivation

- Given: Original image *b*
- Interpret picture as a 2D function:

$$f(x,y): \quad \Omega \rightarrow \Re$$
$$coords \rightarrow intensity$$

- This function *f* should satisfy different conditions over Ω :
  - Similarity with *b*
  - Smoothness

Motivation II

- Design a cost function:

$$I_b(f) = \frac{1}{2}\int_\Omega ((f-b)^2 + \mu|\nabla f|^2)dxdy$$

similarity   smoothness

- Question: how can we find *f* minimizing this cost function?
  => Variational calculus

Variational Calculus

- Real numbers:
  function: $f : \Re \rightarrow \Re$
  mins/maxs: $f'(x_0) := \frac{d}{dx} f(x_0) = 0$
- Variational Calculus:
  functional: $I(f) : f \in C^2 \rightarrow \Re$

Dim reduction through integration

$$I(f) = \int_{x_1}^{x_2} F(x, f, f')dx$$

Ex: $F = ((f-b)^2 + \mu|\nabla f|^2)$

Euler-Langrage

- *F* only depending on *x*, *f*, and first derivative *f'*
- Necessary condition for f_0 as a minimum for *I(f)*:
  Euler-Langrange equation
- Derivation/Analogon:

variation of value

$$\frac{d}{dx} f(x) := \lim_{\epsilon \rightarrow 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

$$\frac{d}{df} I(f,\eta) := \lim_{\epsilon \rightarrow 0} \frac{I(f(x) + \epsilon\eta(x)) - I(f(x))}{\epsilon}$$

variation of function

Extremality Condition

- The condition, that f_0 is a minimum of *I(f)* turns into the condition that $I(\epsilon)$ has a minimum for $\epsilon = 0$
- Hence a variational task becomes a task of finding a minimum:

$$\forall \eta : \quad \frac{d}{d\epsilon} I(f + \eta\epsilon) = 0 \text{ if } \epsilon = 0$$

- In general: difficult, but feasible for us

Euler-Lagrange II

- Insert and calculate:

=> $\int_{x_1}^{x_2} \underbrace{(F_f(x,f,f') - \frac{d}{dx}F_{f'}(x,f,f'))}_{=0,\ \text{since}\ \eta(x_1)=\eta(x_2)=0}\eta(x)dx = 0$

- Apply => **Euler-Langrange equation**

$$F_f(x,f,f') - \frac{d}{dx}F_{f'}(x,f,f') = 0$$

- Differential equation -> can be solved

Recipe

- Define *I(f)*

$$I_b(f) = \frac{1}{2}\int_\Omega \underbrace{((f-b)^2 + \mu|\nabla f|^2)}_{F(x,f,f')}dxdy$$

- Set-Up Euler-Lagrange equation
- **Solve** Euler-Lagrange
- Use boundary conditions to determine unique solution

analytically or numerically

Aufgabe 1

- Task not from image processing
- a) & b): Follow the recipe
- Faithfullness: quadratic deviation

$$\int_a^b (f(x) - h(x))^2 dx$$

- Smoothness: integrated squared gradient

$$\int_a^b ((f'(x))^2 dx$$

- Weighted combination:

$$\int_a^b \underbrace{(f(x) - h(x))^2 + \lambda^{-2}(f'(x))^2}_{F(x,f,f')} dx$$

Aufgabe 1

- b) recipe:

$$f'' = \lambda^2(f-h)$$

„simple" differential equation

- c):
  - Actual *h(x)* given
  - Solution of the Euler-Lagrange given
  - Last step of recipe:
    - use boundary constraints
    - comparison of coefficients
- d): same as a), but not ^2

Aufgabe 2

- Variational computation **with additional condition**

$$\int_a^b G(x,f,f')dx = l \ \ (l = const)$$

- Lagrange (don't confuse with Euler-Lagrange!) => conditions added to min/max function with λ parameter(s)

$$H(x,f,f') = F(x,f,f') + \lambda(G(x,f,f') - l)$$

Aufgabe 2

- Now we solve

$$\int_a^b H(x,f,f',\lambda) = extreme$$

- *H* leads to Euler-Lagrange
- λ can be determined with add. Condition
- λ does not have the same meaning as in A 1)

Aufgabe 2

- a) & b): as Aufgabe 1), adding additional condition
  - Modelling: volume - max, bending – additional condition
  - Euler-Lagrange equation
- c): solve the differential equation analytically, boundary condition for coefficients
- d): determine λ with additional condition
- e) „inverse“:
  - Modelling: volume – condition, bending – max
- f) recipe:
  - Verify: insert in Euler-Lagrange