

# Image Processing and Computer Vision

## Image Processing and Computer Vision

- Processing of continuous images
  - linear filtering
  - Fourier transformation
- Wiener filtering
- Nonlinear diffusion

## Computer Vision

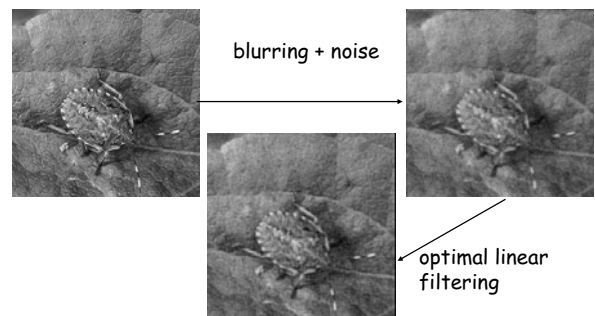
What is computer vision? **interpreting images!**



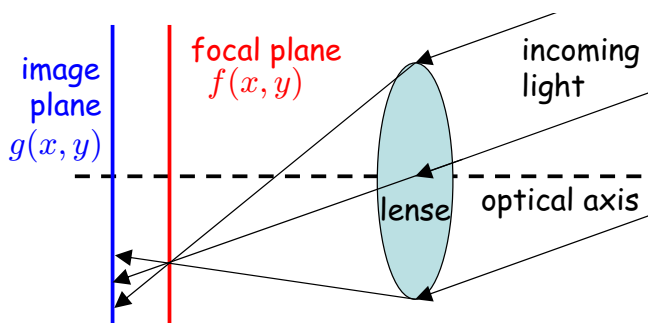
The computer sees 1001110100101010000000001110101...

## Image Processing

What is image processing? **restoring images without extraction of semantic information!**



## The Image Formation Process



## Mathematical Modelling of Image Processing

**Def.:** An image is a continuous, two-dimensional function of the light intensity

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}_+ \\ (x, y) \mapsto f(x, y)$$

**Question:** How can we compensate an image deformation, e.g., defocussing?

**Goal:** reconstruct  $f(x, y)$  from  $g(x, y)$  in the presence of noise!

**Model assumption:**

- 1) When  $f(x, y)$  is shifted then  $g(x, y)$  is shifted as well.
- 2) Doubling the incoming light intensity will double the brightness  $g(x, y)$ .

## Linear Shift-Invariant Systems

**Strategy for restauration:** invert the transformation which has mapped the original image  $f(x, y)$  to the defocussed image  $g(x, y)$ .

**Linearity:** (assumption)

$$f_1 \longrightarrow \boxed{\text{transform}} \longrightarrow g_1$$

$$f_2 \longrightarrow \boxed{\text{transform}} \longrightarrow g_2$$

$$\alpha f_1 + \beta f_2 \longrightarrow \boxed{\text{transform}} \longrightarrow \alpha g_1 + \beta g_2 \quad \forall \alpha, \beta \in \mathbb{R}$$

- Linearity is typically only in the low intensity range fulfilled since physical systems tend to saturate.
- $f_i, g_i$  are intensities  $\equiv$  power per area with  $f_i, g_i \geq 0$  in the full domain.
- Often we experience non-linear imaging errors!

**Shift invariance:** (assumption)

$$f(x, y) \longrightarrow \boxed{\text{transform}} \longrightarrow g(x, y)$$

$$f(x - a, y - b) \longrightarrow \boxed{\text{transform}} \longrightarrow g(x - a, y - b)$$

- Shift invariance holds only in a limited range since images are finite objects.

**Remarks:** The assumption of linearity is a significant limitation but it gives the advantage that the linear filter theory is completely developed.

- An analogous one-dimensional theory applies to passive electrical circuits, although there time is the essential dimension and causality constraints the signal.

## How Can We Identify a Transformation?

**Dirac's  $\delta$ -function (1D):**  $\int_{-\infty}^{\infty} \delta(x - a) f(x) dx = f(a)$

- Integration with the  $\delta$ -function "samples" the function  $f(x)$  at the position  $x_0 = a$ .
- The  $\delta$ -function is a "generalized function".
- Regularization:

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \begin{cases} \frac{1}{\epsilon} & |x| \leq \frac{\epsilon}{2} \\ 0 & \text{else} \end{cases}$$

or

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{2\pi}\epsilon} \exp\left(-\frac{x^2}{2\epsilon^2}\right)$$

## Convolution and the Point Spread Function

Assumption:  $\delta(x, y) \longrightarrow \boxed{\mathcal{T}} \longrightarrow h(x, y)$

With linearity and shift invariance it holds:

$$\begin{aligned} g(x, y) &= \mathcal{T} f(x, y) \\ &= \mathcal{T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta(x - \xi, y - \eta) d\xi d\eta \\ &\stackrel{\text{linearity}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \underbrace{[\mathcal{T} \delta(x - \xi, y - \eta)]}_{\substack{h(x - \xi, y - \eta) \\ \text{[shift inv.]}}} d\xi d\eta \\ &= (f * h)(x, y) \end{aligned}$$

Linear, shift invariant systems can be written as **convolutions!**

## Identification of the Kernel

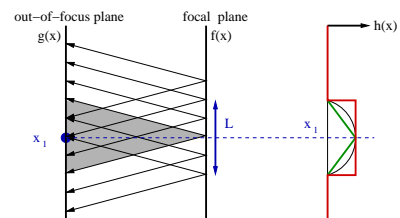
Let  $f(x, y) = \delta(x, y)$ , i.e., the image is a white dot with "infinite" intensity. Then the measured image  $g(x, y)$  is given by

$$\begin{aligned} g(x, y) &= (\delta * h)(x, y) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\xi, \eta) h(x - \xi, y - \eta) d\xi d\eta \\ &= h(x, y) \end{aligned}$$

$$\Rightarrow \mathcal{T} \delta(x, y) = h(x, y)$$

$\Rightarrow$  testing the linear shift-invariant system with a  $\delta$ -peak will reveal the **convolution kernel**  $h(x, y)$  of the system.

## Schematic View of a Convolution

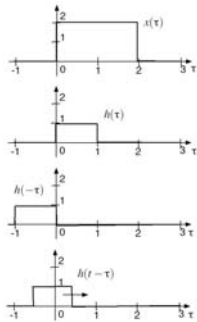


- $g(x_1)$  depends on  $f(x)$  for all  $x \in [x_1 - \frac{L}{2}, x_1 + \frac{L}{2}]$ .
- convolution kernel  $h_{x_1}(x)$  describes the influence of  $f(x)$  onto  $g(x_1)$ .
- shift invariance of  $h_{x_1}(x)$  results in cumulative influence:

$$\begin{aligned} g(x_1) &= \int_{-L/2}^{L/2} f(x) h(x_1 - x) dx = \int_{-L/2}^{L/2} f(x_1 - x) h(x) dx \\ &\approx f(0)h(x_1)\Delta + f(\Delta)h(x_1 - \Delta)\Delta + f(2\Delta)h(x_1 - 2\Delta)\Delta + \dots \\ &\quad + f(-\Delta)h(x_1 + \Delta)\Delta + f(-2\Delta)h(x_1 + 2\Delta)\Delta + \dots \end{aligned}$$

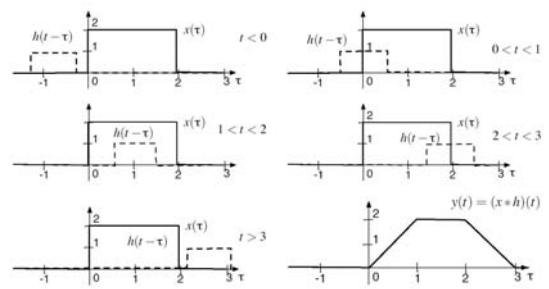
## Convolution: 1D-Example

$$y(t) = (x * h)(t) = \int x(\tau)h(t - \tau)d\tau$$



## Convolution: 1D-Example (cont'd)

$$y(t) = (x * h)(t) = \int x(\tau)h(t - \tau)d\tau$$

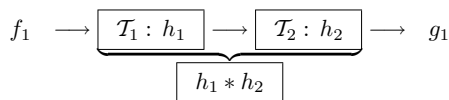


## Facts about Convolution

- Linear shift-invariant (LSI) systems can be written as convolutions.
- The convolution kernel  $h$  characterizes the LSI system uniquely.
- Cascades of LSI systems: the convolution is commutative and associative:

$$g * h = h * g$$

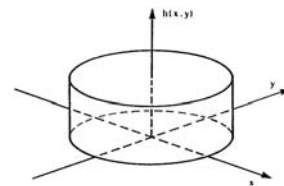
$$(f * g) * h = f * (g * h)$$



⇒ one of the most important operations in signal processing

## Convolution Kernel for Image Defocussing

Defocussing an image amounts to convolving it with a 'pillbox':



$$h(x, y) = \begin{cases} \frac{1}{\pi R^2} & x^2 + y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases}$$

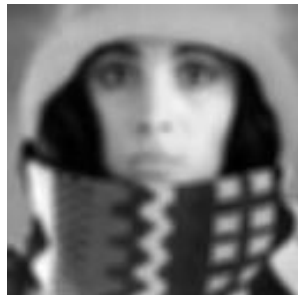
Note: this convolution kernel is normalized:  $\int \int h(x, y) dx dy = 1$

## Convolution Kernel for Image Defocussing

original image



convolved with pillbox kernel



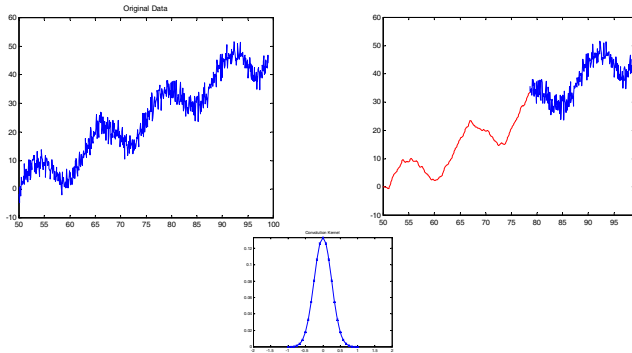
## A Motion Kernel

Each light dot is transformed into a short line along the  $x$ -axis:

$$h(x, y) = \frac{1}{2l} [\theta(x + l) - \theta(x - l)] \delta(y)$$



## Denoising Time Series



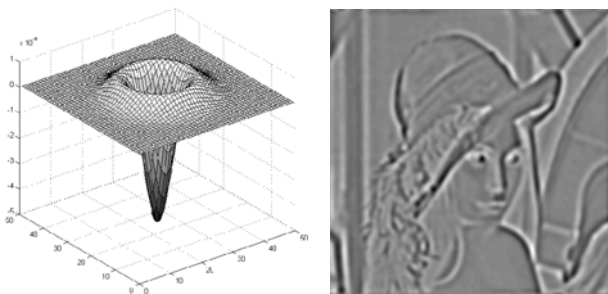
## Lena with Gaussian Blurring and Noise



Gaussian blurring kernel:

$$h(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

## Lena Convolved with a Laplacian Filter



Laplacian filter:

$$h(x, y) = \nabla^2 \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

Note: here the normalization is  $\int \int h(x, y) dx dy = 0$ .

## The Fourier Transformation

**Def.:** Let  $f$  be an absolutely integrable function over  $\mathbb{R}$ . The Fourier transformation of  $f$  is defined as

$$\hat{f}(u) \equiv \mathcal{F}[f(x)] = \int_{-\infty}^{+\infty} f(x) \exp(-i2\pi ux) dx.$$

The inverse Fourier transformation is given by the formula

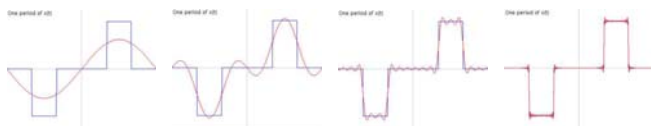
$$f(x) \equiv \mathcal{F}^{-1}[\hat{f}(u)] = \int_{-\infty}^{+\infty} \hat{f}(u) \exp(i2\pi ux) du.$$

**Note:** while  $f(x)$  is always real,  $\hat{f}(u)$  is typically complex.

- $\hat{f}(u)$  is also called the **continuous spectrum** of  $f(x)$ .
- If  $x$  is a space coordinate, then  $u$  is called the **spatial frequency**.

**Inversion formula:**  $f(x)$  is represented as a continuous superposition of waves with amplitude  $\hat{f}(u)$ .

**Example** of an odd function approximated by sinus waves  
(Remember:  $\exp(ix) = \cos(x) + i \sin(x)$ ):



$$f(x) \approx \hat{f}(u_0) \sin(2\pi u_0 x) + \hat{f}(u_1) \sin(2\pi u_1 x) + \hat{f}(u_2) \sin(2\pi u_2 x) + \dots$$

## Fourier Transformation: Example 1 (box)

Given the box function

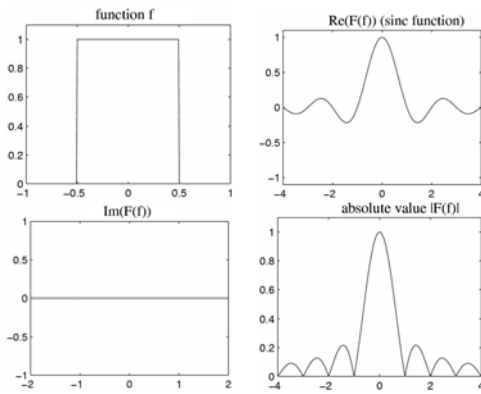
$$f(x) = \frac{1}{2l} (\theta(x+l) - \theta(x-l)) = \begin{cases} \frac{1}{2l} & \text{if } |x| \leq l \\ 0 & \text{otherwise} \end{cases}$$

the Fourier transform is

$$\begin{aligned} \hat{f}(u) \equiv \mathcal{F}[f(x)] &= \int_{-\infty}^{+\infty} f(x) \exp(-i2\pi ux) dx \\ &= \int_{-l}^l \frac{1}{2l} \cdot (\cos(2\pi ux) - \underbrace{i \sin(2\pi ux)}_{f \rightarrow 0}) dx \\ &= \frac{\sin(2\pi ul)}{2\pi ul} \equiv \text{sinc}(2\pi ul) \end{aligned}$$

## Fourier Transformation: Example 1 (box)

Graphs of box and sinc-function for  $l = \frac{1}{2}$ :



## Fourier Transformation: Example 2 (Gauss)

Given the function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left(-\frac{x^2}{2\sigma_x^2}\right)$$

the Fourier transform is

$$\begin{aligned} \hat{f}(u) &\equiv \mathcal{F}[f(x)] = \int_{-\infty}^{+\infty} f(x) \exp(-i2\pi ux) dx \\ &= \frac{1}{\sqrt{2\pi}\sigma_x} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2\sigma_x^2}\right) \cdot \underbrace{(\cos(2\pi ux) - i \sin(2\pi ux))}_{f \rightarrow 0} dx \\ &= \dagger \exp\left(-\frac{u^2}{2\sigma_u^2}\right) \quad \text{where } \sigma_u = \frac{1}{2\pi\sigma_x} \end{aligned}$$

<sup>†</sup> [Abramowitz, Stegun: Handbook of Mathematical Functions, 1972]

⇒ the Fourier transform of a Gaussian is a (unnormalized) Gaussian!

The larger the variance  $\sigma_x^2$ , the smaller the variance  $\sigma_u^2$ :  $\sigma_x \cdot \sigma_u = \frac{1}{2\pi}$

## Fourier Transformation: Example 3 (Dirac's $\delta$ )

The Fourier transform of Dirac's  $\delta$ -function is

$$\begin{aligned} \hat{\delta}(u) &\equiv \mathcal{F}[\delta(x)] = \int_{-\infty}^{+\infty} \delta(x) \exp(-i2\pi ux) dx \\ &= \exp(-i2\pi u \cdot 0) \\ &= 1 \end{aligned}$$

⇒ the Fourier transform of the  $\delta$ -function equals 1 for *all* frequencies  $u$ .

## Properties of the Fourier Transformation

**Linearity:** If  $\mathcal{F}[f(x)] = \hat{f}(u)$  and  $\mathcal{F}[g(x)] = \hat{g}(u)$  then it holds for all complex numbers  $a, b \in \mathbb{C}$

$$\mathcal{F}[af(x) + bg(x)] = a\hat{f}(u) + b\hat{g}(u)$$

**Shift:** If  $\mathcal{F}[f(x)] = \hat{f}(u)$  then it holds for  $c \in \mathbb{R}$

$$\mathcal{F}[f(x - c)] = \hat{f}(u) \exp(-i2\pi cu)$$

**Modulation:** If  $\mathcal{F}[f(x)] = \hat{f}(u)$  then it holds for  $c \in \mathbb{R}$

$$\mathcal{F}[f(x) \exp(i2\pi cx)] = \hat{f}(u - c)$$

**Scaling:** If  $\mathcal{F}[f(x)] = \hat{f}(u)$  and  $c > 0$

$$\mathcal{F}[f(cx)] = \frac{1}{c} \hat{f}\left(\frac{u}{c}\right)$$

**Differentiation:** Let  $f$  be piecewise continuous and absolutely integrable. If the function  $xf(x)$  is absolutely integrable then the Fourier transform  $\hat{f}$  is continuous and differentiable. It holds

$$\begin{aligned} \mathcal{F}[xf(x)] &= \frac{i}{2\pi} \frac{d}{du} \hat{f}(u) \\ \mathcal{F}\left[\frac{d}{dx} f(x)\right] &= i2\pi u \hat{f}(u) \end{aligned}$$

**Parseval's Equality:** Let  $f$  be piecewise continuous and absolutely integrable. Then the Fourier transform  $\hat{f}(u) = \mathcal{F}[f(x)]$  satisfies:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |\hat{f}(u)|^2 du$$

**Power Spectrum:** Considering the auto-correlation function  $\Phi_{ff}(x)$  of a complex function  $f$  for  $x \in \mathbb{R}$ ,

$$\Phi_{ff}(x) = \int_{-\infty}^{+\infty} \bar{f}(\xi - x) f(\xi) d\xi.$$

The Fourier transform is given by

$$\hat{\Phi}_{ff}(u) \equiv \mathcal{F}[\Phi_{ff}(x)] = |\hat{f}(u)|^2.$$

( $\bar{f}(x)$  is the conjugate complex function of  $f(x)$ )

## Fourier Transform of Convolution

**Given:** convolution  $g(x) = (f * h)(x) = \int f(\xi)h(x - \xi)d\xi$

Calculate Fourier transform of  $g$ :

$$\begin{aligned}\hat{g}(u) &\equiv \mathcal{F}[g(x)] = \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{+\infty} f(\xi)h(x - \xi)d\xi \right] \exp(-i2\pi ux) dx \\ &= \int_{-\infty}^{+\infty} f(\xi) \left[ \int_{-\infty}^{+\infty} h(x - \xi) \exp(-i2\pi ux) dx \right] d\xi \\ &= \int_{-\infty}^{+\infty} \hat{h}(u) f(\xi) \exp(-i2\pi u\xi) d\xi \\ &= \hat{h}(u) \hat{f}(u)\end{aligned}$$

⇒ **Convolution** in spatial domain becomes **multiplication** in Fourier space.

## Modulation Transfer Function

**System Behavior in Fourier Space:** How is a harmonic oscillation transformed by convolution kernel  $h$ ?

⇒ amplitude modulation  $A(u)$ :

$$\exp(i2\pi ux) \longrightarrow \boxed{\text{kernel } h(x)} \longrightarrow A(u) \exp(i2\pi ux)$$

**Eigenfunction of the convolution** with eigenvalue  $A(u)$  is the oscillation  $f(x) = \exp(i2\pi ux)$ .

$$\begin{aligned}\text{Output } g(x) &= (f * h)(x) = \int \exp(i2\pi u\xi)h(x - \xi)d\xi \\ &= \exp(i2\pi ux) \int \exp(-i2\pi u\xi)h(\xi)d\xi \\ &= \hat{h}(u) \exp(i2\pi ux)\end{aligned}$$

**Note:** the eigenvalue  $A(u)$  equals  $\hat{h}(u) = \mathcal{F}[h](u)$ .

## Image Filtering in the Frequency Domain

**2D Fourier transformation** of an image  $f(x, y)$ :

$$\hat{f}(u, v) \equiv \mathcal{F}[f(x, y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) \exp(-i2\pi(ux + vy)) dx dy$$

**High-pass filtering:** remove low frequencies, for example choose maximum value  $B$ :

$$\hat{f}_{\text{hp}}(u, v) = \begin{cases} \hat{f}(u, v) & \text{if } u^2 + v^2 > B^2 \\ 0 & \text{otherwise} \end{cases}$$

**Inverse Fourier transformation** yields high-pass-filtered image

$$f_{\text{hp}}(x, y) = \mathcal{F}^{-1}[\hat{f}_{\text{hp}}(u, v)]$$

## Example of Image Filtering



original image



high-pass-filtered

⇒ edge detection

**Low-pass filtering:** analogous to high-pass filter, but remove high frequencies

**Example:**



original image



low-pass-filtered

⇒ removing noise

## The Image Restoration Problem

$$f(x, y) \longrightarrow \boxed{h(x, y)} \longrightarrow g(x, y) \longrightarrow \boxed{\tilde{h}(x, y)} \longrightarrow f(x, y)$$

The 'inverse' kernel  $\tilde{h}(x, y)$  should compensate the effect of the image degradation  $h(x, y)$ , i.e.,

$$(\tilde{h} * h)(x, y) = \delta(x, y)$$

$\tilde{h}$  may be determined more easily in Fourier space:

$$\mathcal{F}[\tilde{h}](u, v) \cdot \mathcal{F}[h](u, v) = 1$$

To determine  $\mathcal{F}[\tilde{h}]$  we need to estimate

1. the distortion model  $h(x, y)$  (point spread function) or  $\mathcal{F}[h](u, v)$  (modulation transfer function)
2. the parameters of  $h(x, y)$ , e.g.  $r$  for defocussing.

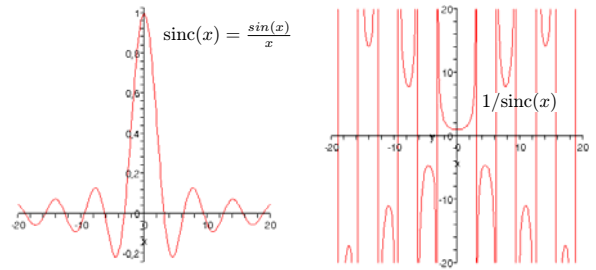
## Image Restoration: Example

**Example: motion blur**  $h(x, y) = \frac{1}{2l}(\theta(x+l) - \theta(x-l))\delta(y)$

(a light dot is transformed into a small line in  $x$  direction).

**Fourier transformation:**

$$\begin{aligned} \mathcal{F}[h](u, v) &= \frac{1}{2l} \int_{-l}^{+l} \exp(-i2\pi ux) \underbrace{\int_{-\infty}^{+\infty} \delta(y) \exp(-i2\pi vy) dy}_{=1} dx \\ &= \frac{\sin(2\pi ul)}{2\pi ul} =: \text{sinc}(2\pi ul) \end{aligned}$$



$$\hat{h}(u) = \mathcal{F}[h](u) = \text{sinc}(2\pi ul)$$

$$\mathcal{F}[\tilde{h}](u) = 1/\hat{h}(u)$$

**Problems:**

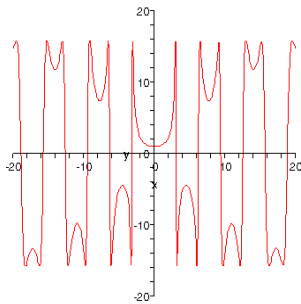
- Convolution with the kernel  $h$  completely cancels the frequencies  $\frac{\nu}{2l}$  for  $\nu \in \mathcal{Z}$ . Frequencies which disappear cannot be recovered!
- Noise amplification for  $\mathcal{F}[h](u, v) \ll 1$ .

## Avoiding Noise Amplification

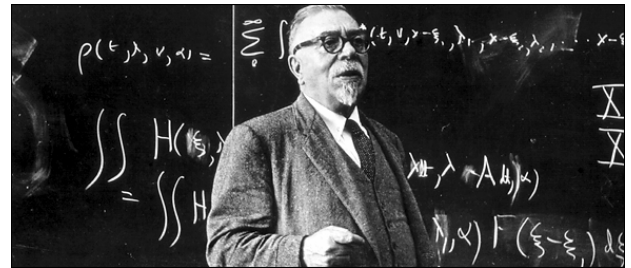
**Regularized reconstruction filter:**

$$\tilde{\mathcal{F}}[\tilde{h}](u, v) = \frac{\mathcal{F}[h]}{|\mathcal{F}[h]|^2 + \epsilon^2}$$

Singularities are avoided by the regularization  $\epsilon^2$ .



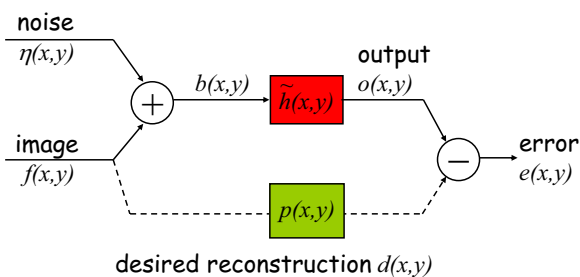
## The Wiener Filter: Optimal Linear Filtering for Noise Suppression and Image Reconstruction



**American mathematician** who developed the theory of Brownian motion;  $\Rightarrow$  Wiener measure, numerical PDE solutions, [linear filter theory](#).

**Cybernetics** as the new science for systems design and control. Norbert Wiener broke new ground in robotics, computer control, and automation.

## Optimal Linear Filtering and Noise Suppression



**Given:**  $b(x, y) = f(x, y) + \eta(x, y)$   
image = signal + noise

**Goal:** Reconstruct signal  $f(x, y)$  as "good" as possible.

**Noise model:** we assume that the signal and the noise are uncorrelated, i.e. the cross-correlation is zero:

$$\Phi_{f\eta}(a, b) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f}(x-a, y-b) \eta(x, y) dx dy = 0.$$

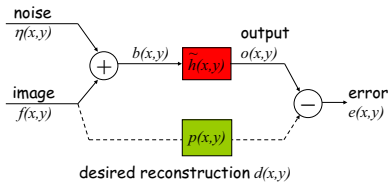
**Task:** find  $o(x, y)$  which reconstructs the original image  $f(x, y)$  *as good as possible* from the observed image  $b(x, y)$ !  
(setting  $d(x, y) = f(x, y)$  with  $p(x, y)$  being the identity map)

In some situations the desired reconstruction  $d(x, y)$  might differ from  $f(x, y)$  since we might prefer a smoothed or sharpened version (given by the transformation  $p(x, y)$ ) of the original image.

**Assumption:** use a *linear filter*  $\tilde{h}(x, y)$  for reconstruction, i.e.,

$$o(x, y) = (b * \tilde{h})(x, y).$$

## Goal of the Wiener Filter



### Quality measure for image restoration:

What does “as good as possible” actually mean?

⇒ **average quadratic error**

$$\text{error} = E = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (o(x, y) - d(x, y))^2 dx dy$$

**Goal:** find the kernel  $\tilde{h}$  that minimizes this error.

## Derivation of the Wiener Filter

**Error decomposition:**  $(\mathbf{x} := (x, y)^T)$

$$\begin{aligned} E &= \int_{\Omega} (o(\mathbf{x}) - d(\mathbf{x}))^2 d\mathbf{x} = \int_{\Omega} (o^2 - 2od + d^2) d\mathbf{x} \\ &= \underbrace{\int_{\Omega} o(\mathbf{x})^2 d\mathbf{x}}_{(1)} - 2 \underbrace{\int_{\Omega} o(\mathbf{x})d(\mathbf{x}) d\mathbf{x}}_{(2)} + \underbrace{\int_{\Omega} d(\mathbf{x})^2 d\mathbf{x}}_{(3)} \end{aligned}$$

⇒ simplify each of the three integrals:

**integral (3):**  $\int_{\Omega} d(\mathbf{x})^2 d\mathbf{x} = \Phi_{dd}(0, 0)$

where  $\Phi_{dd}(0, 0)$  is  $d$ 's auto-correlation with no displacement.

**integral (2):** inserting  $o(\mathbf{x}) = (b * \tilde{h})(\mathbf{x})$  yields

$$\begin{aligned} \int_{\Omega} o(\mathbf{x})d(\mathbf{x})d\mathbf{x} &= \int_{\Omega} \left[ \int_{\Omega} b(\mathbf{x} - \boldsymbol{\xi})\tilde{h}(\boldsymbol{\xi}) d\boldsymbol{\xi} \right] d(\mathbf{x}) d\mathbf{x} \\ &= \int_{\Omega} \underbrace{\left[ \int_{\Omega} b(\mathbf{x} - \boldsymbol{\xi})d(\mathbf{x}) d\mathbf{x} \right]}_{\Phi_{bd}(\boldsymbol{\xi})} \tilde{h}(\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \int_{\Omega} \Phi_{bd}(\boldsymbol{\xi})\tilde{h}(\boldsymbol{\xi})d\boldsymbol{\xi} \end{aligned}$$

where  $\Phi_{bd}(\boldsymbol{\xi})$  is the cross-correlation of  $b$  and  $d$  with displacement  $\boldsymbol{\xi} = (\xi_1, \xi_2)^T$ .

**integral (1):** inserting  $o(\mathbf{x}) = (b * \tilde{h})(\mathbf{x})$  yields

$$\begin{aligned} \int_{\Omega} o^2 d\mathbf{x} &= \int_{\Omega} \left( (b * \tilde{h})(\mathbf{x}) \right)^2 d\mathbf{x} \\ &= \int_{\Omega} \left( \int_{\Omega} \int_{\Omega} b(\mathbf{x} - \boldsymbol{\xi})b(\mathbf{x} - \boldsymbol{\alpha})\tilde{h}(\boldsymbol{\xi})\tilde{h}(\boldsymbol{\alpha}) d\boldsymbol{\alpha} d\boldsymbol{\xi} \right) d\mathbf{x} \\ &\stackrel{\mathbf{x}' = \mathbf{x} - \boldsymbol{\alpha}}{=} \int_{\Omega} \int_{\Omega} \underbrace{\left[ \int_{\Omega} b(\mathbf{x}' - \boldsymbol{\xi} + \boldsymbol{\alpha})b(\mathbf{x}') d\mathbf{x}' \right]}_{\Phi_{bb}(\boldsymbol{\xi} - \boldsymbol{\alpha})} \tilde{h}(\boldsymbol{\xi})\tilde{h}(\boldsymbol{\alpha}) d\boldsymbol{\alpha} d\boldsymbol{\xi} \\ &= \int_{\Omega} \int_{\Omega} \Phi_{bb}(\boldsymbol{\xi} - \boldsymbol{\alpha})\tilde{h}(\boldsymbol{\xi})\tilde{h}(\boldsymbol{\alpha}) d\boldsymbol{\alpha} d\boldsymbol{\xi} \end{aligned}$$

The first term in the average error defines a quadratic form of the auto-correlation  $\Phi_{bb}(\boldsymbol{\xi} - \boldsymbol{\alpha})$  with  $\tilde{h}$ .

## Wiener Filter Defined by Correlations

The average quadratic error can now be rewritten in terms of various auto/cross correlations:

$$\begin{aligned} E(\tilde{h}) &= \underbrace{\int_{\Omega} \int_{\Omega} \Phi_{bb}(\boldsymbol{\xi} - \boldsymbol{\alpha})\tilde{h}(\boldsymbol{\xi})\tilde{h}(\boldsymbol{\alpha}) d\boldsymbol{\alpha} d\boldsymbol{\xi}}_{(1)} \\ &\quad - 2 \underbrace{\int_{\Omega} \Phi_{bd}(\boldsymbol{\xi})\tilde{h}(\boldsymbol{\xi}) d\boldsymbol{\xi}}_{(2)} + \underbrace{\Phi_{dd}(0, 0)}_{(3)} \end{aligned}$$

To minimize  $E(\tilde{h})$  w.r.t. the reconstructing filter  $\tilde{h}$  is a problem of variational calculus, e.g.,  $\min_{\tilde{h}} \int f(\tilde{h}(\mathbf{x}))d\mathbf{x}$ .

## Variation of the Wiener Filter

Next, we find the filter  $\tilde{h}$  that minimizes the error function  $E(\tilde{h})$ , using the *variational calculus*:

- we assume that the kernel  $\tilde{h}(x, y)$  minimizes  $E(\tilde{h})$ .
- we choose an *arbitrary* function  $\delta\tilde{h}(x, y)$ ;  
( $\delta\tilde{h}(x, y) = 0$  on the boundary of the image)
- then  $\tilde{h}(x, y) + \epsilon \cdot \delta\tilde{h}(x, y)$  is also a valid kernel ( $\epsilon \geq 0$ ).
- **Minimality Condition:** since  $\tilde{h}(x, y)$  minimizes  $E(\tilde{h})$ , it has to be a minimum of  $E(\tilde{h})$  with the condition:

$$\left. \frac{\partial}{\partial \epsilon} E(\tilde{h} + \epsilon \cdot \delta\tilde{h}) \right|_{\epsilon=0} = 0 \quad \forall \delta\tilde{h}(x, y) \in \mathcal{C}^0$$



Replace  $\tilde{h}$  by  $\tilde{h} + \epsilon \cdot \delta\tilde{h}$  to obtain  $E(\tilde{h} + \epsilon \cdot \delta\tilde{h})$ :

$$\begin{aligned}
 E(\tilde{h} + \epsilon \cdot \delta\tilde{h}) &= \int_{\Omega} \int_{\Omega} \Phi_{bb}(\xi - \alpha) (\tilde{h}(\xi) + \epsilon \delta\tilde{h}(\xi)) \times \\
 &\quad (\tilde{h}(\alpha) + \epsilon \delta\tilde{h}(\alpha)) d\alpha d\xi \\
 &\quad - 2 \int_{\Omega} \Phi_{bd}(\xi) (\tilde{h}(\xi) + \epsilon \delta\tilde{h}(\xi)) d\xi + \Phi_{dd}(0, 0) \\
 &= E(\tilde{h}) + 2\epsilon \int_{\Omega} \int_{\Omega} \underbrace{\Phi_{bb}(\xi - \alpha) \tilde{h}(\alpha)}_{(\Phi_{bb} * \tilde{h})(\xi)} d\alpha d\xi \\
 &\quad - 2\epsilon \int_{\Omega} \Phi_{bd}(\xi) \delta\tilde{h}(\xi, \eta) d\xi + \mathcal{O}(\epsilon^2) \\
 \Rightarrow \frac{\partial}{\partial \epsilon} E(\tilde{h} + \epsilon \cdot \delta\tilde{h}) \Big|_{\epsilon=0} &= -2 \int_{\Omega} (\Phi_{bd}(\xi) - (\Phi_{bb} * \tilde{h})(\xi)) \delta\tilde{h}(\xi) d\xi
 \end{aligned}$$

Since  $\delta\tilde{h}(\xi)$  is an arbitrary function, the equation

$$\frac{\partial}{\partial \epsilon} E(\tilde{h} + \epsilon \cdot \delta\tilde{h}) \Big|_{\epsilon=0} = 0$$

requires the integrand  $(\Phi_{bd} - \Phi_{bb} * \tilde{h})$  to vanish for all values  $\mathbf{x} = (x, y)^T$  (fundamental theorem of variational calculus):

$$\Phi_{bd}(\mathbf{x}) = (\Phi_{bb} * \tilde{h})(\mathbf{x}) \quad \text{Wiener-Hopf equation}$$

The convolution kernel (point spread function)  $\tilde{h}(\mathbf{x})$  of the optimal linear filter has to satisfy the Wiener-Hopf equation.

## Fourier Analysis of the Wiener Filter

In Fourier space the Wiener-Hopf equation yields:  
 ( $\hat{f} := \mathcal{F}[f]$  denotes the Fourier transform)

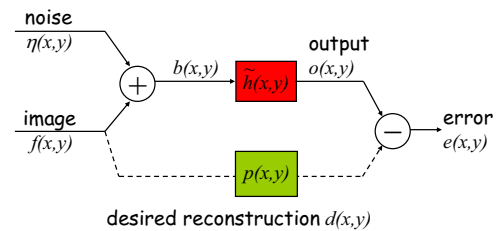
$$\begin{aligned}
 \hat{\Phi}_{bd} &= \hat{\Phi}_{bb} \cdot \mathcal{F}[\tilde{h}] \\
 \mathcal{F}[\tilde{h}](u, v) &= \frac{\hat{\Phi}_{bd}(u, v)}{\hat{\Phi}_{bb}(u, v)} = \frac{\hat{\Phi}_{fd}(u, v)}{\hat{\Phi}_{ff}(u, v) + \hat{\Phi}_{\eta\eta}(u, v)}
 \end{aligned}$$

The last equality holds because we assumed that

- $b(x, y) = f(x, y) + \eta(x, y)$ ,
- the noise  $\eta$  is **not** correlated with the signal  $f$ :  $\Phi_{f\eta}(x, y) = 0$  for all  $x, y$ .

$$\Rightarrow \hat{\Phi}_{bb} = \hat{\Phi}_{f+\eta, f+\eta} = \hat{\Phi}_{ff} + \underbrace{\hat{\Phi}_{f\eta}}_{=0} + \underbrace{\hat{\Phi}_{\eta f}}_{=0} + \hat{\Phi}_{\eta\eta}$$

## Wiener Filter: Improving a Noisy Image



If  $d = f$ , the Fourier transform of the optimal linear filter for the (unknown) original signal  $f$  is

$$\mathcal{F}[\tilde{h}](u, v) = \frac{\hat{\Phi}_{ff}(u, v)}{\hat{\Phi}_{ff}(u, v) + \hat{\Phi}_{\eta\eta}(u, v)} = \frac{1}{1 + \frac{\hat{\Phi}_{\eta\eta}(u, v)}{\hat{\Phi}_{ff}(u, v)}}$$

## Signal-to-Noise Ratio

**Definition:** the ratio

$$\text{SNR}(u, v) = \frac{\hat{\Phi}_{ff}(u, v)}{\hat{\Phi}_{\eta\eta}(u, v)}$$

is called the *signal-to-noise ratio* (at the frequencies  $(u, v)$ ).

**SNR**( $u, v$ ) **large:** the filter behaves almost like the identity map.

**SNR**( $u, v$ ) **small:** the filter is proportional to the SNR.  
 $\Rightarrow$  damping.

## Statistics of Natural Images

**Observation [Fields, 1987]:** the power spectrum of natural images  $f(x, v)$  decays as

$$\begin{aligned}
 \hat{\Phi}_{ff}(u, v) &= \hat{\Phi}_{ff}(\rho, \theta) \propto \frac{1}{\rho^2} \\
 \Rightarrow \hat{\Phi}_{ff}(\rho) &= \int \hat{\Phi}_{ff}(\rho, \theta) \rho d\theta \propto \int \frac{1}{\rho^2} \rho d\theta \propto \frac{1}{\rho}
 \end{aligned}$$

**Note:** in the Fourier space, the polar coordinates  $\rho, \theta$  are used in place of the Cartesian coordinates  $u, v$  (frequencies).

**Assumption concerning noise:** the noise is spatially uncorrelated, i.e.,

$$\Phi_{\eta\eta}(x, y) = \Phi_0 \cdot \delta(x, y)$$

$$\Rightarrow \text{Fourier transform: } \hat{\Phi}_{\eta\eta}(u, v) = \Phi_0$$

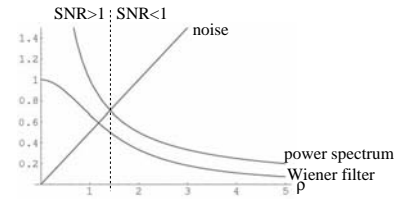
$$\Rightarrow \text{polar coordinates: } \hat{\Phi}_{\eta\eta}(\rho) = \int \hat{\Phi}_{\eta\eta}(\rho, \theta) \rho d\theta \\ = \int \Phi_0 \rho d\theta \propto \Phi_0 \cdot \rho$$

## Wiener Filter: Improving Noisy Natural Images

**natural image:**

- power spectrum:  $\hat{\Phi}_{ff}(\rho) \propto \frac{1}{\rho}$

- noise:  $\hat{\Phi}_{\eta\eta}(\rho) \propto \Phi_0 \cdot \rho$



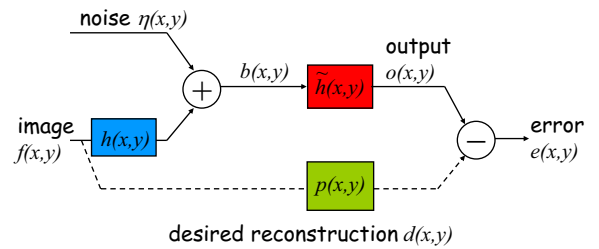
**Wiener filter:**  $\mathcal{F}[\tilde{h}](\rho) = \left(1 + \frac{\hat{\Phi}_{\eta\eta}(\rho)}{\hat{\Phi}_{ff}(\rho)}\right)^{-1}$

**Two limiting cases:**

- SNR  $\gg 1 \Rightarrow \mathcal{F}[\tilde{h}] \approx 1$   
... no modulation of the low frequencies
- SNR  $\ll 1 \Rightarrow \mathcal{F}[\tilde{h}] \approx \hat{\Phi}_{ff}/\hat{\Phi}_{\eta\eta} \propto 1/\rho^2$   
... damping of the high frequencies

## Optimal Linear Filtering for Image Reconstruction with Simultaneous Noise Suppression

**Assumption:** There exists a "degradation kernel"  $h$  which has transformed the image before noise perturbation!



**Given:**  $b(x, y) = (f * h)(x, y) + \eta(x, y)$   
image = signal  $f$  & + noise  
degradation  $h$

**Noise:** we assume that the signal and the noise are uncorrelated:  $\Phi_{f\eta} = 0$ .

**Task:** reconstruct  $f(x, y)$  as good as possible from  $b(x, y)$ !

**Assumption:** use a linear filter  $\tilde{h}(x, y)$  to compensate the degradation and to filter out the noise, i.e.,

$$o(x, y) = (b * \tilde{h})(x, y)$$

## Derivation of Reconstruction Wiener Filter

**Autocorrelation of image  $b(x, y)$ :**

$$\hat{\Phi}_{bb} = \hat{\Phi}_{f * h + \eta, f * h + \eta} = \hat{\Phi}_{f * h, f * h} + 2\hat{\Phi}_{f * h, \eta} + \hat{\Phi}_{\eta\eta} \\ = \hat{h}^2 \hat{\Phi}_{ff} + \hat{h} \underbrace{\hat{\Phi}_{f\eta}}_{=0} + \hat{h} \underbrace{\hat{\Phi}_{\eta f}}_{=0} + \hat{\Phi}_{\eta\eta},$$

since a correlation of a convolution  $f * h$  with a function  $g$  is the convolution of the correlation  $f * g$  with the kernel  $h$ .

**Result in Fourier space:**

$$\hat{\Phi}_{bd} = \hat{\Phi}_{bb} \cdot \mathcal{F}[\tilde{h}] \quad \dots \text{as before} \\ \mathcal{F}[\tilde{h}](u, v) = \frac{\hat{\Phi}_{bd}(u, v)}{\hat{\Phi}_{bb}(u, v)} = \frac{\hat{h}(u, v) \cdot \hat{\Phi}_{fd}(u, v)}{\hat{h}^2(u, v) \cdot \hat{\Phi}_{ff}(u, v) + \hat{\Phi}_{\eta\eta}(u, v)}$$

**Assumption:**  $d(x, y) = f(x, y)$ , i.e., the desired image is the original one:

$$\mathcal{F}[\tilde{h}](u, v) = \frac{\hat{h}(u, v)}{\hat{h}^2(u, v) + \frac{\hat{\Phi}_{\eta\eta}(u, v)}{\hat{\Phi}_{ff}(u, v)}} = 1/\text{SNR}(u, v)$$

**Note:** this filter corresponds to the heuristic regularization for avoiding noise amplification (slide 39):  $\epsilon^2 = 1/\text{SNR}(u, v)$

**Two limiting cases:**

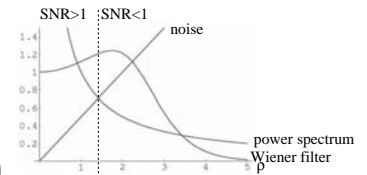
- $\text{SNR} \gg 1 \Rightarrow \mathcal{F}[\tilde{h}](u, v) \approx \frac{1}{\hat{h}(u, v)}$   
... cf. direct derivation of image restoration kernel (slide 36)
- $\text{SNR} \ll 1 \Rightarrow \mathcal{F}[\tilde{h}] \approx \hat{h}(u, v) \hat{\Phi}_{ff} / \hat{\Phi}_{\eta\eta} \propto 1/\rho^2$   
... in natural images  $\Rightarrow$  damping of high frequencies.

## Wiener Filter: Sharpening and Denoising Natural Images

**Assume:** blurring with Gaussian kernel (in polar coordinates):

$$h(r) \propto \exp\left(-\frac{r^2}{2\sigma_r^2}\right)$$

$$\Rightarrow \hat{h}(\rho) \propto \exp\left(-\frac{\rho^2}{2\sigma_\rho^2}\right)$$



$\Rightarrow$  **Wiener filter:**

$$\mathcal{F}[\tilde{h}](\rho) \propto \frac{\exp\left(-\frac{\rho^2}{2\sigma_\rho^2}\right)}{\left(\exp\left(-\frac{\rho^2}{2\sigma_\rho^2}\right)\right)^2 + \frac{\hat{\Phi}_{\eta\eta}(\rho)}{\hat{\Phi}_{ff}(\rho)}} \propto \hat{\Phi}_0 \cdot \rho^2 \text{ in natural images}$$

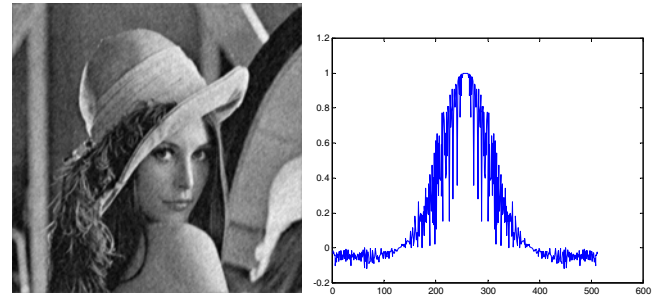
## Lena: original & noisy (PSNR=7.2)



Mean Square Error:  $\text{MSE}(f, g) = \frac{1}{\Omega} \int_{\Omega} (f(\mathbf{x}) - g(\mathbf{x}))^2 d\mathbf{x}$

peak SNR:  $\text{PSNR}(f, g) := 20 \log_{10} \left( \frac{255}{\sqrt{\text{MSE}}} \right)$

## Optimal Linear Filter (Lena PSNR=7.2)



## Lena: noisy image & reconstruction



The Lena image has been blurred with a Gaussian kernel of  $\sigma_{\text{kernel}} = 3$  and it has been degraded with Gaussian noise ( $\sigma_{\text{noise}} = 30$ ).  
Image quality: PSNR=7.2

Peak SNR of reconstruction: PSNR=24.7

## Reconstruction Filter

