Variational Methods for Image Enhancement

Goal: find a smoothing image transformation according to some optimality criterion (cf. Wiener filter)

Model assumptions:

- the filtered image f should be similar to the original image b
- the filtered image f should be smooth

Continuous formulation: given $b : \Omega \to \mathbb{R}$, determine $f : \Omega \to \mathbb{R}$ such that it minimizes the cost

$$I_b(f) = \frac{1}{2} \int_\Omega \Big(\underbrace{(f-b)^2}_{\text{similarity}} + \mu \underbrace{|\nabla f|^2}_{\text{smoothness}} \Big)\, dx\, dy$$

The parameter μ is called 'regularization parameter'.

Question: How can the minimizing function f of the cost $I_b(f)$ be obtained?

Excursion: Calculus of Variations

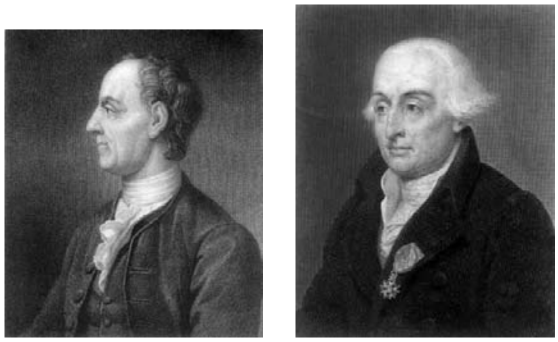
The Calculus of Variations

Calculus of Real Numbers:

- considers real-valued *functions* $f(x)$ that map *real numbers* $x \in \mathbb{R}$ to real numbers
- if x_0 is a minimum of f, then x_0 necessarily satisfies $f'(x_0) := \frac{df}{dx}(x_0) = 0$
- x_0 is a unique minimum if f is strictly convex

Variational Calculus:

- considers real-valued *functionals* $I(f)$ that map *functions* $f \in \mathcal{C}^2$ to real numbers
- if f_0 is a minimum of I, then f_0 necessarily satisfies the corresponding *Euler-Lagrange equation*, a differential equation in f
- f_0 is a unique minimum if I is strictly convex



The mathematicians **Leonhard Euler** (left, 1707–1783) and **Joseph-Louis Lagrange** (right, 1736–1813) are two of the founders of the calculus of variations (Source: http://www-gap.dcs.st-and.ac.uk/~history/).

Euler-Lagrange Equation in 1D

Goal: determine a smooth function $f \in \mathcal{C}^2[x_1, x_2]$ which minimizes the functional

$$I(f) = \int_{x_1}^{x_2} F(x, f, f')\, dx$$

under the boundary conditions $f(x_1) = f_1$ and $f(x_2) = f_2$.

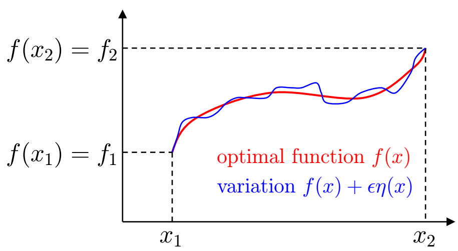
Euler-Lagrange equation: necessary condition for the minimizing function:

$$F_f - \frac{d}{dx} F_{f'} = 0$$

where we use the abbreviations

$$F_f = \frac{\partial}{\partial f} F(x, f, f') \qquad F_{f'} = \frac{\partial}{\partial f'} F(x, f, f')$$

Derivation of the Euler-Lagrange Equation



Assumption: let the function $f(x)$ be a minimum of I.

Idea: we add an arbitrary perturbation function $\eta \in \mathcal{C}^2[x_1, x_2]$ with $\eta(x_1) = \eta(x_2) = 0$ with a scaled amplitude ϵ to the function $f(x)$. This small variation $\epsilon\eta(x)$ should not change the value of the functional "too much".

Variation of $f(x)$**:** $g(x) := f(x) + \epsilon\eta(x)$
with the derivative $g'(x) = f'(x) + \epsilon\eta'(x)$

(note that the boundary constraints $g(x_1) = f_1$ and $g(x_2) = f_2$ are also fulfilled for g due to $\eta(x_1) = 0$ and $\eta(x_2) = 0$)

Necessary condition of extremality:

$$\forall\,\eta: \quad \frac{d}{d\epsilon} I(g)\Big|_{\epsilon=0} = 0$$

(since $\phi(\epsilon) := I(g)$ has a minimum in $\epsilon = 0$, so $\phi'(0) = 0$.)

Strategy of the analysis: exchange differentiation and integration and apply the chain rule to compute the total derivative of $F(x, g, g')$ with respect to ϵ:

$$\begin{aligned}
0 &= \frac{d}{d\epsilon} I(g)\Big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\int_{x_1}^{x_2} F(x,g,g')\,dx\Big|_{\epsilon=0} \\
&= \int_{x_1}^{x_2}\left(\frac{d}{d\epsilon}F(x,g,g')\right)dx\Big|_{\epsilon=0} \\
&= \int_{x_1}^{x_2} F_f(x,g,g')\eta(x) + F_{f'}(x,g,g')\eta'(x)\,dx\Big|_{\epsilon=0} \\
&= \int_{x_1}^{x_2} F_f(x,f,f')\eta(x) + F_{f'}(x,f,f')\eta'(x)\,dx
\end{aligned}$$

Partial integration of the second term:
($\int_a^b u \cdot v'\,dx = [u \cdot v]_a^b - \int_a^b u' \cdot v\,dx$)

$$\int_{x_1}^{x_2} F_{f'}(x,f,f')\eta'(x)\,dx =$$

$$\underbrace{\Big[F_{f'}(x,f,f')\eta(x)\Big]_{x_1}^{x_2}}_{=0,\text{ since } \eta(x_1)=\eta(x_2)=0} - \int_{x_1}^{x_2}\frac{d}{dx}\left(F_{f'}(x,f,f')\right)\eta(x)\,dx$$

Inserting into the necessary condition yields:

$$\int_{x_1}^{x_2}\left(F_f(x,f,f') - \frac{d}{dx}F_{f'}(x,f,f')\right)\eta(x)\,dx = 0$$

which has to hold for all variations $\eta \in \mathcal{C}^2[x_1,x_2]$ with $\eta(x_1) = \eta(x_2) = 0$.

Fundamental lemma of variational calculus: If

$$\int_a^b g(x)h(x)\,dx = 0$$

holds for all $h \in \mathcal{C}^2[a,b]$ with $h(a) = h(b) = 0$, then $g(x) \equiv 0$.

Applying this lemma yields the Euler-Lagrange equation:

$$F_f(x,f,f') - \frac{d}{dx}F_{f'}(x,f,f') = 0$$

□

Natural boundary conditions

If explicit boundary constraints $f(x_1) = f_1$ and $f(x_2) = f_2$ are not given for f, it is possible to deduce the following 'natural' constraints from the variational formulation of the problem:

$$F_{f'}(x,f,f') = 0$$

for the boundary points $x = x_1$ and $x = x_2$.

Note that a sufficient number of boundary constraints is necessary to find a *unique* solution for a differental equation.

Explicit Form: What is $\frac{d}{dx}F_{f'}$?

$\frac{d}{dx}$ is the *total derivative* of the functional $F_{f'}$, i.e.

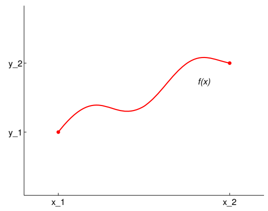
$$\begin{aligned}
\frac{d}{dx}F_{f'} &= \frac{\partial}{\partial x}F_{f'}(x,f,f') + \frac{\partial}{\partial f}F_{f'}(x,f,f')f' + \frac{\partial}{\partial f'}F_{f'}f'' \\
&= F_{f',x} + F_{f',f}f' + F_{f',f'}f''
\end{aligned}$$

Euler-Lagrange equation in explicit form:

$$\begin{aligned}
0 &= F_f - \frac{d}{dx}F_{f'} \\
&= F_f - F_{f',x} - F_{f',f}f' - F_{f',f'}f''
\end{aligned}$$

Example: Curve of minimal length

Goal: find the function f of shortest length connecting two points (x_1, y_1) and (x_2, y_2)



Curve length of f is given by:

$$I(f) = \int_{x_1}^{x_2} \sqrt{1 + (f')^2}\, dx$$

Partial derivatives of the integrand $F(x, f, f') = \sqrt{1 + (f')^2}$:

$$F_f = 0, \qquad F_{f'} = \frac{f'}{\sqrt{1 + (f')^2}}$$

Euler-Lagrange equation in this case:

$$\frac{d}{dx} \frac{f'(x)}{\sqrt{1 + (f'(x))^2}} = 0 \quad \Longleftrightarrow \quad \frac{f'(x)}{\sqrt{1 + (f'(x))^2}} = c \in \mathbb{R}$$

Solve for f':

$$f'(x) = \frac{c}{\sqrt{1 - c^2}} \quad \Longleftrightarrow \quad f(x) = \frac{c}{\sqrt{1 - c^2}} x + d$$

$\Rightarrow$ f is a straight line, values of c and d are determined by the boundary conditions $f(x_1) = y_1, f(x_2) = y_2$

Variational Calculus with Constraints

Isoperimetric Problem:

$$\min_f \; I(f) \; = \int_{x_1}^{x_2} F(x, f, f')\, dx$$

$$\text{s.t.} \qquad 0 \; = \int_{x_1}^{x_2} G_j(x, f, f')\, dx \quad 1 \le j \le m$$

Introduce Lagrange variables:

$$\tilde{F}(x, f, f') = F(x, f, f') + \sum_j \lambda_j G_j(x, f, f')$$

Euler-Lagrange equation in this case:

$$\tilde{F}_f - \frac{d}{dx} \tilde{F}_{f'} = 0$$

Choose λ_j such that the constraints are fulfilled.

Potential Extension: Higher Order Derivatives

Integrand with higher order derivatives:

$$I(f) = \int_{x_1}^{x_2} F(x, f, f', f'', \dots)\, dx$$

Euler-Lagrange equation in this case:

$$F_f - \frac{d}{dx} F_{f'} + \frac{d^2}{dx^2} F_{f''} - \dots = 0$$

Note that the alternating sign comes from iterated partial integration.

Potential Extension: Dependence on Several Functions

Integrand with dependence on the functions $f_1, f_2, \dots$:

$$I(f_1, f_2, \dots) = \int_{x_1}^{x_2} F(x, f_1, f_2, \dots, f_1', f_2', \dots)\, dx$$

Euler-Lagrange equations in this case:

$$\begin{aligned} F_{f_1} - \frac{d}{dx} F_{f_1'} &= 0 \\ F_{f_2} - \frac{d}{dx} F_{f_2'} &= 0 \\ &\dots \end{aligned}$$

We derive as many equations as we have functional dependencies.

Two Dimensional Variational Calculus

Functional is an integral in higher dimensions:

$$I(f) = \int_\Omega F(x, y, f, f_x, f_y)\, dx\, dy$$

with partial derivatives: $f_x := \frac{\partial f}{\partial x}, \; f_y := \frac{\partial f}{\partial y}$

Boundary constraints: the values of $f(x, y)$ are given on the boundary $\partial\Omega$ of the region Ω.

Euler-Lagrange equation for the 2-D case:

$$F_f - \frac{\partial}{\partial x} F_{f_x} - \frac{\partial}{\partial y} F_{f_y} = 0$$

Can be derived similarly to the 1-D case based on small variations $\epsilon\eta$ and application of Green's integral theorem.

Natural boundary conditions: if n denotes the function giving the normal vector for every point on the boundary $\partial\Omega$, we obtain the constraint

$$n^\top \begin{pmatrix} F_{f_x} \\ F_{f_y} \end{pmatrix} = 0$$

on the boundary $\partial\Omega$, or equivalently

$$F_{f_x}\frac{dy}{ds} = F_{f_y}\frac{dx}{ds}$$

where s is a parameter for the boundary curve.

Application: Variational Methods for Image Enhancement

Original problem: find smoothing image transformation f which minimizes the cost

$$I_b(f) = \frac{1}{2}\int_\Omega \Big(\underbrace{(f-b)^2}_{\text{similarity}} + \mu \underbrace{|\nabla f|^2}_{\text{smoothness}} \Big)\, dx\, dy$$

Partial derivatives of the integrand $F(x, y, f, f_x, f_y) = \frac{1}{2}(f-b)^2 + \frac{\mu}{2}(f_x^2 + f_y^2)$:

$$F_f = f - b, \qquad F_{f_x} = \mu f_x, \qquad F_{f_y} = \mu f_y$$

Euler-Lagrange equation in this case:

$$\begin{aligned} 0 &= F_f - \frac{\partial}{\partial x}F_{f_x} - \frac{\partial}{\partial y}F_{f_y} \\ &= f - b - \frac{\partial}{\partial x}(\mu f_x) - \frac{\partial}{\partial y}(\mu f_y) \\ &= f - b - \mu \underbrace{f_{xx} + f_{yy}}_{\Delta f} \end{aligned}$$

- As it contains partial derivatives of the unknown function $f(x, y)$, this is a *partial differential equation (PDE)*.
- Such equations usually have to be solved numerically.
- Discretization via finite difference approximation leads to linear system of equations which can be solved iteratively (e.g. Jacobi method).

Natural boundary conditions $n^\top \begin{pmatrix} F_{f_x} \\ F_{f_y} \end{pmatrix} = 0$ on the image boundary $\partial\Omega$ give

$$0 = n^\top \nabla f = \partial_n f$$

where $\partial_n f$ denotes the derivative of f in the direction of n.

- The normal derivative has to vanish at the image boundaries.
- Numerically, this can be established by extending the image by mirroring the boundary pixels.

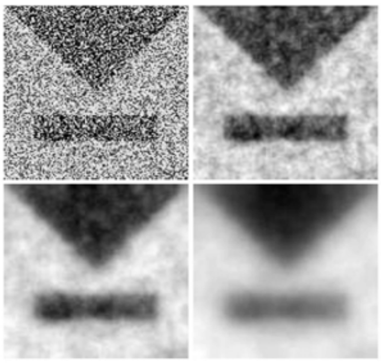
Connection to linear diffusion: Euler-Lagrange equation

$$f_{xx} + f_{yy} + \frac{b-f}{\mu} = 0$$

can be interpreted as steady-state ($t \to \infty$) of linear diffusion with an additional bias term

$$f_t = f_{xx} + f_{yy} + \frac{b-f}{\mu}.$$

$\Rightarrow$ discretization of linear diffusion process gives a gradient descent method for minimizing $I_b(f)$



Top left: Test image, 128×128 pixels. **Top right:** Variational method with $\mu = 5$. **Bottom left:** $\mu = 20$. **Bottom right:** $\mu = 100$. Author: J. Weickert.

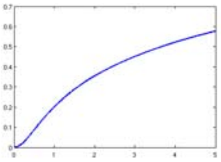
Variational Calculus and Nonlinear Diffusion

Nonlinear diffusion reduces blurring of edges

Idea: replace smoothness term $|\nabla f|^2$ by potential function $\Psi(|\nabla f|)$ which penalizes large gradients less severely

Perona-Malik potential:

$$\Psi(|\nabla f|) = \frac{\lambda^2}{2} \log\left(1 + \frac{|\nabla f|^2}{\lambda^2}\right)$$



Cost minimization with Perona-Malik potential (no similarity term):

$$I(f) := \int_\Omega \Psi(|\nabla f|)\, dx\, dy = \int_\Omega \frac{\lambda^2}{2} \log\left(1 + \frac{|\nabla f|^2}{\lambda^2}\right) dx\, dy$$

Partial derivatives of $\Psi(|\nabla f|)$:

$$\Psi_f = 0, \qquad \Psi_{f_x} = \frac{f_x}{1 + |\nabla f|^2/\lambda^2}, \qquad \Psi_{f_y} = \frac{f_y}{1 + |\nabla f|^2/\lambda^2}$$

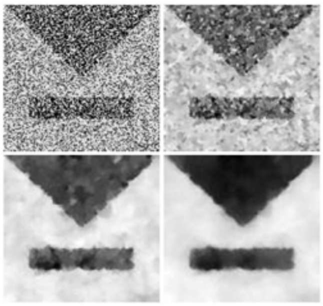
Euler-Lagrange equation:

$$\frac{\partial}{\partial x}\Psi_{f_x} + \frac{\partial}{\partial y}\Psi_{f_y} - \Psi_f = \text{div}\left(\frac{1}{1 + |\nabla f|^2/\lambda^2}\nabla f\right) = 0 \approx f_t$$

$\Rightarrow$ diffusion process defines gradient descent method for minimizing $I(f)$.

Nonlinear Variational Method

Cost minimization with potential $\Psi(|\nabla f|) = \lambda\sqrt{1 + |\nabla f|^2/\lambda^2}$



Top left: Test image, 128×128 pixels. **Top right:** Nonlinear variat. method with $\lambda = 1$ and $\mu = 20$. **Bottom left:** $\mu = 50$. **Bottom right:** $\mu = 100$. Author: J. Weickert.