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Solve before: May 30, 2006

Visual Computing

4) Convolution and Fourier Transform (from Mod. & Sim. exam 05/06)

Consider the one-dimensional box function f

$$f(x) := \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

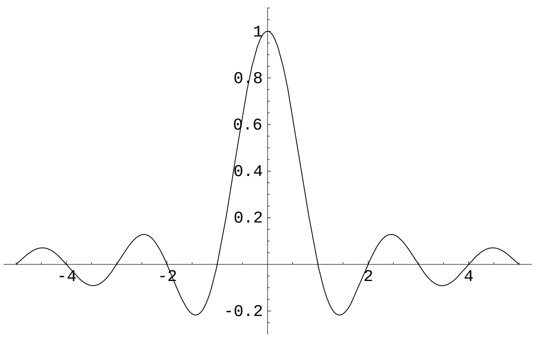
a) Calculate the Fourier transform of the function $f(x)$.

Solution:

$$\begin{aligned} \hat{f}(u) \equiv \mathcal{F}[f(x)] &= \int_{-\infty}^{+\infty} f(x) \exp(-\imath 2\pi ux)\, dx \\ &= \int_{-0.5}^{0.5} 1 \cdot (\cos(2\pi ux) - \imath \underbrace{\sin(2\pi ux)}_{\int \to 0})\, dx \\ &= \frac{\sin(\pi u)}{\pi u} \equiv \operatorname{sinc}(\pi u) \end{aligned}$$

b) Assume the filter f is applied to a signal $s(x)$: $[f * s](x)$. Which frequencies in the spectrum of s will be lost? Which part of the spectrum will be damped the most: low, medium or high frequency bands?

Solution: The following graph depicts $\hat{f}(u)$*:*



As $\hat{f}(u) = 0$ *at* $u = \pm 1, \pm 2, \ldots$ *these frequencies are erased in the filtered image.*

Considering the envelope of the Fourier transformation of f*, one directly observes that high frequency bands are damped the most.*

c) Iterative convolution (central limit theorem):

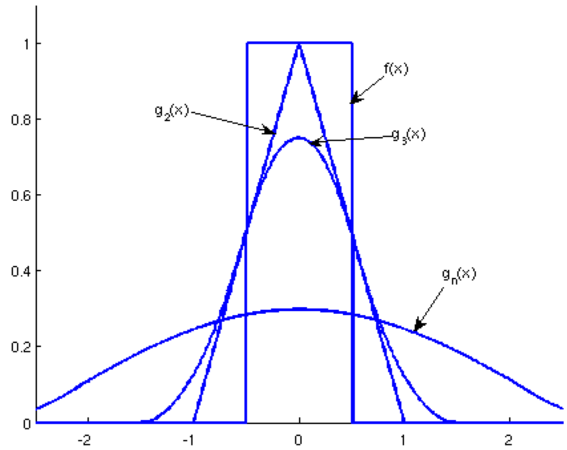
- Calculate the Fourier transform of the function $g_2(x) := [f * f](x)$.
Solution:

$$\begin{aligned} \hat{g}_2(u) \equiv \mathcal{F}[g_2(x)] &= \mathcal{F}[f * f(x)] \\ &= \mathcal{F}[f(x)] \cdot \mathcal{F}[f(x)] \\ &= \frac{\sin(\pi u)}{\pi u} \cdot \frac{\sin(\pi u)}{\pi u} = \operatorname{sinc}^2(\pi u) \end{aligned}$$

- Calculate the Fourier transform of the function g_n resulting from convolving n versions of f, $g_n(x) := [\underbrace{f * \cdots * f}_{n \text{ times}}](x)$.
Solution:

$$\hat{g}_n(u) \equiv \mathcal{F}[g_n(x)] = \frac{\sin^n(\pi u)}{\pi^n u^n} = \operatorname{sinc}^n(\pi u)$$

- Which filter function is obtained for $n \to \infty$? Draw a qualitative sketch of f, g_2, and g_3 and observe the convergence.
Solution:
Direct convolution in spatial domain indicates the following convergence:



Convolution of two box filters results in a tent filter (piecewise linear). Subsequent convolutions then yield piecewise quadratic, piecewise cubic, etc. filters, which in the limit converge to a Gaussian filter.

d) Assume the filter $g_{\lim} := \lim_{n\to\infty} g_n$ is applied to a signal $s(x)$: $[g_{\lim} * s](x)$. Which frequencies in the spectrum of s will be damped the most: low, medium, or high ones? Which frequencies will be erased completely?

Solution: As $g_{\lim}$ corresponds to a Gaussian filter, high frequencies are damped the most, while no frequencies are erased completely.